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Article

On l_p-Valued Functions for Henstock-Kurzweil-Stieltjes-◊-Double Integrals on Time Scales

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Abstract

We define Henstock-Kurzweil-Stieltjes-◊-double integral for l_p-valued functions on time scales. Furthermore, some of the basic properties of this integral with values ranging in an l_p-space for $0 < p < 1$ are proved and discussed. Our first theorem guarantees the uniqueness of Henstock-Kurzweil-Stieltjes-◊-double integral of an l_p-valued functions. The Henstock-Kurzweil-Stieltjes-◊-double integrals generalizes the Henstock integral for l_p-valued functions in literature.

Keywords: Double integral, Henstock-Kurzweil integral, l_p-space, Time scales.

1. Introduction

Henstock [3] in a note published in 1963 introduced an integral for real-valed function. This integral is usually referred to as Hentock integral, and seems to be easily handled than that of Riemann and Lebesgue. In the last decades or more, Henstock integral has been well studied by many of the researchers investigating certain single real-valued functions on time scales. The following references are recommended for further study of Henstock integral for single real-valued functions and other important properties [2,3,5]. The theory of time scales and its wide applications to so many branches of Mathematics was introduced by Stephen Hilger [4]. In 2006, Peterson and Thompson [7] introduced Henstock delta integral on time scales and gave some of its basic properties. In 2008, Thompson [8] used covering arguments to study Henstock-Kurzweil integrals for single real valued-functions on time scale. Cao [2] considered Henstock integral for Banach-valued functions.

In a recent paper, the authors [1] gave a time scale fuzzy Henstock-Kurzweil Stieltjes integral involving two functions. Furthermore, in the same paper [1], the authors gave some basic properties of fuzzy Henstock-Kurzweil-Stieltjes-◊-double integral on time scale and also proved the uniform, monotone and dominated convergence theorems for fuzzy Henstock-Kurzweil-Stieltjes-◊-double integral on time scale. The aim of this paper is to introduce the l_p-valued functions for Henstock-Kurzweil-Stieltjes-◊-double integral on time scales and obtain some basic properties of l_p-valued functions for Henstock-Kurzweil-Stieltjes-◊-double integral.

2. Preliminaries

Let X be an underlying space, F denotes the σ-algebra of measurable sets, μ the measure, and (X, F, μ) denotes a σ-finite measure space. If $1 \leq p < \infty$ 1 is the space $l_p(X, F, \mu)$ consists of all complex-valued measurable functions on X satisfying

$$\int_X |f(x)|^p d\mu(x) < \infty.$$

We simply write l_p when the underlying measure space has been specified. If $f \in l_p$, the l_p norm of f is defined by

$$\|f\|_p = \left(\int_X |f(x)|^p d\mu(x)\right)^{\frac{1}{p}}.$$

The case where $p = 1$, then the space $l_1(X, F, \mu)$ consists of all integrable functions on X, also the case $p = 2$ is a Hilbert space. The spaces l_p are examples of normed linear spaces. The basic property of satisfied by the norm is the triangle inequality. In most applications of l_p spaces, the range of p which is often of interest is $1 \leq p < \infty$. When $0 < p < 1$, the function $\|.\|_{l_p}$ does not satisfy the triangle inequality and for such p, the space l_p has no non-trivial bounded linear functional. When $p = 1$, the norm $\|.\|_{l_1}$ satisfies the triangle inequality, and l_1 is a complete normed linear space.

The following are examples of l_p spaces:

(i). Let $X = \mathbb{R}^d$ and μ a Lebesgue measure. Then,

$$\|f\|_p = \left(\int_{\mathbb{R}^d} |f(x)|^p d\mu(x)\right)^{\frac{1}{p}}.$$

(ii). Let $X = \mathbb{Z}$, and μ a counting measure. Then,

$$\|f\|_p = \left(\sum_{n=-\infty}^{\infty} |a_n|^p\right)^{\frac{1}{p}},$$

which is the discrete version of the l_p space.

Now, we discuss the following concepts in $\mathbb{T}_1 \times \mathbb{T}_2$. Let $a, b \in \mathbb{T}_1, c, d \in \mathbb{T}_2$, where $a < d, c < d$, and $\mathcal{R} = [a, b)_{\mathbb{T}_1} \times [c, d)_{\mathbb{T}_2} = \{(t, s): t \in [a, b), s \in [c, d), t \in \mathbb{T}_1, s \in \mathbb{T}_2\}$. Let $g_1, g_2: \mathbb{T}_1 \times \mathbb{T}_2 \to \mathbb{R}$ be monotone increasing functions on $[a, b]_{\mathbb{T}_1}$ and $[c, d]_{\mathbb{T}_2}$, and $F: \mathbb{T}_1 \times \mathbb{T}_2 \to \mathbb{R}$ a bounded function on $\mathcal{R}$. Let P_1 and P_2 be partitions of $[a, b]_{\mathbb{T}_1}$ and $[c, d]_{\mathbb{T}_2}$ such that $P_1 = \{t_0, t_1, \ldots, t_n\} \subset [a, b]_{\mathbb{T}_1}$ and $P_2 = \{s_0, s_1, \ldots, s_n\} \subset [c, d]_{\mathbb{T}_2}$. Let $\{\xi_1, \xi_2, \ldots, \xi_n\}$ represent an arbitrary chosen points from $[a, b]_{\mathbb{T}_1}$ with $\xi_i \in [t_{i-1}, t_i)_{\mathbb{T}_1}, i = 1,2, \ldots, n$. Also, let $\{\zeta_1, \zeta_2, \ldots, \zeta_n\}$ represent an arbitrary chosen points from $[c, d]_{\mathbb{T}_2}$ with $\zeta_j \in [s_{j-1}, s_j)_{\mathbb{T}_2}, j = 1,2, \ldots, k$, see [6].

Definition 2.1. *A pair $\delta = (\delta_L, \delta_R)$ of real-valued functions defined on $[a, b]_{\mathbb{T}_1} \times [c, d]_{\mathbb{T}_2}$ is said to be a Δ-gauge for $[a, b]_{\mathbb{T}_1} \times [c, d]_{\mathbb{T}_2}$ if*

$$\begin{cases} \delta_L(t, s) > 0 \quad \text{on} \quad (a, b]_{\mathbb{T}_1} \times (c, d]_{\mathbb{T}_2} \\ \delta_R(t, s) > 0 \;\text{ on }\; [a, b)_{\mathbb{T}_1} \times [c, d)_{\mathbb{T}_2} \\ \delta_L(a) \geq 0,\; \delta_R(b) \geq 0 \;\; \delta_L(c) \geq 0 \;\; \delta_R(d) \geq 0 \\ \delta_R(t, s) \geq \sigma(t, s) - (t, s) \quad \text{forall } \; t \in [a, b)_{\mathbb{T}_1}, s \in [c, d)_{\mathbb{T}_2}. \end{cases}$$

A pair $\gamma = (\gamma_L, \gamma_R)$ of real-valued functions defined on $[a, b]_{\mathbb{T}_1} \times [c, d]_{\mathbb{T}_2}$ is said to be a ∇-gauge for $[a, b]_{\mathbb{T}_1} \times [c, d]_{\mathbb{T}_2}$ if

$$\begin{cases} \gamma_L(t, s) > 0 \quad \text{on} \quad (a, b]_{\mathbb{T}_1} \times (c, d]_{\mathbb{T}_2} \\ \gamma_R(t, s) > 0 \;\text{ on }\; [a, b)_{\mathbb{T}_1} \times [c, d)_{\mathbb{T}_2} \\ \gamma_L(a) \geq 0,\; \gamma_R(b) \geq 0,\; \gamma_L(c) \geq 0,\; \gamma_R(d) \geq 0 \\ \gamma_R(t, s) \geq (t, s) - \rho(t, s) \quad \text{forall } \; t \in (a, b]_{\mathbb{T}_1}, s \in (c, d]_{\mathbb{T}_2}. \end{cases}$$

Given a Δ-gauge δ and a ∇-gauge γ, the partitions $P_1 = \{t_0, t_1, \ldots, t_n\} \subset [a, b]_{\mathbb{T}_1}$ with tag points $\xi_i \in [t_{i-1}, t_i]_{\mathbb{T}_1}$ and $P_2 = \{s_0, s_1, \ldots, s_k\} \subset [c, d]_{\mathbb{T}_2}$ with tag points $\zeta_j \in [s_{j-1}, s_j]_{\mathbb{T}_2},\; j = 1,2, \ldots, k$, is said to be:

- δ-fine if $g(\xi_i, \zeta_j) - \delta_L \leq g(t_{i-1}, s_{j-1}) < g(t_i, s_j) \leq g((\xi_i, \zeta_j) + \delta_R(\xi_i, \zeta_j))$ and

- γ-fine if $g(\xi_i,\zeta_j)-\gamma_L \le g(t_{i-1},s_{j-1}) < g(t_i,s_j) \le g((\xi_i,\zeta_j)+\gamma_R(\xi_i,\zeta_j))$.

Now, we present newly the following definitions.

Definition 2.2. [1] *Let $F:[a,b)_{\mathbb{T}_1}\times[c,d)_{\mathbb{T}_2}\to l_p$ be an l_p-valued function on $\mathcal{R}$ and let g be a monotone increasing function defined on $[a,b)_{\mathbb{T}_1}\times[c,d)_{\mathbb{T}_2}$ with partitions $P_1=\{t_0,t_1,\ldots,t_n\}\subset[a,b]_{\mathbb{T}_1}$ with tag points $\xi_i\in[t_{i-1},t_i]_{\mathbb{T}_1}$ for $i=1,2,\ldots,n$ and $P_2=\{s_0,s_1,\ldots,s_k\}\subset[c,d]_{\mathbb{T}_2}$ with tag points $\zeta_j\in[s_{j-1},s_j]_{\mathbb{T}_2}$ for $j=1,2,\ldots,k$. Then*

$$S(P_1,P_2,F,g_1,g_2)=\sum_{i=1}^{n}\sum_{j=1}^{k}F(\xi_i,\zeta_j)(g_1(t_i)-g_1(t_{i-1}))(g_2(s_j)-g_2(s_{j-1}))$$

is defined as l_p-valued Henstock-Kurweil-Stieltjes-⋄-double sum of F with respect to functions g_1 and g_2.

Let $P=P_1\times P_2$ and $\Diamond\, g_{1_i}\,\Diamond\, g_{2_j}=(g_1(t_i)-g_1(t_{i-1}))(g_2(s_j)-g_2(s_{j-1}))$, then the Henstock-Kurweil-Stieltjes-◊-double sum of F with respect to functions g_1 and g_2 is denoted by $S(P,F,g)$ is written as

$$S(P,F,g_1,g_2)=\sum_{i=1}^{n}\sum_{j=1}^{k}F(\xi_i,\zeta_j)\,\Diamond\, g_{1_i}\,\Diamond\, g_{2_j},\ \ (i=1,\ldots,n;\ \ j=1,\ldots,k).$$

Definition 2.3. [1] *Let $F:[a,b)_{\mathbb{T}_1}\times[c,d)_{\mathbb{T}_2}\to l_p$ be an l_p-valued function on $\mathcal{R}=[a,b)_{\mathbb{T}_1}\times[c,d)_{\mathbb{T}_2}$. We say that F is l_p-valued Henstock-Kurzweil-Stieltjes-◊-integrable with respect to monotone increasing functions $g_1:[a,b)_{\mathbb{T}_1}\longrightarrow\mathbb{R}$ and $g_2:[c,d)_{\mathbb{T}_2}\longrightarrow\mathbb{R}$ if there exists $I\in l_p$ such that for every $\varepsilon>0$, there are ◊-gauges δ_1 and δ_2 (or γ_1 and γ_2) for $[a,b)_{\mathbb{T}_1}$ and $[c,d)_{\mathbb{T}_2}$ respectively such that*

$$\parallel S(P,F,g)-I\parallel_p<\varepsilon$$

provided that $P_1=\{t_0,t_1,\ldots,t_n\}\subset[a,b]_{\mathbb{T}_1}$ with tag points $\xi_i\in[t_{i-1},t_i]_{\mathbb{T}_1}$ for $i=1,\ldots,n$ is a δ_1-fine (or γ_1) and $P_2=\{s_0,s_1,\ldots,s_k\}\subset[c,d]_{\mathbb{T}_2}$ with tag points $\zeta_j\in[s_{j-1},s_j]_{\mathbb{T}_2}$, $j=1,2,\ldots,k$ is a δ_2-fine (or γ_2) are δ-fine (or γ) partitions of $[a,b)_{\mathbb{T}_1}$ and $[c,d)_{\mathbb{T}_2}$ respectively.

We say that I is the l_p-valued Henstock-Kurzweil-Stieltjes-◊-double integral of F with respect to g_1 and g_2 defined on $[a,b)_{\mathbb{T}_1}\times[c,d)_{\mathbb{T}_2}$, and write

$$\int\int_{\mathcal{R}}F(t,s)\,\Diamond\, g_1(t)\,\Diamond\, g_2(s)=I.$$

3. Main Results

Our first theorem guarantees the uniqueness of Henstock-Kurzweil-Stieltjes-◊-double integral of an l_p-valued function, if it exists.

Theorem 3.1. *If $F:(a,b]_{\mathbb{T}_1}\times(c,d]_{\mathbb{T}_2}\to l_p$ is Henstock-Kurzweil-Stieltjes-⋄-double integrable with respect to monotone increasing functions g_1,g_2 on $(a,b]_{\mathbb{T}_1}\times(c,d]_{\mathbb{T}_2}$, then the Henstock-Kurzweil-Stieltjes-◊-double integral of F is unique.*

Proof. Suppose that I_1 and I_2 are both Henstock-Kurzweil-Stieltjes-◊-double integrals of F on $[a,b)_{\mathbb{T}_1}\times[c,d)_{\mathbb{T}_2}$. With the assumption that I_1 and I_2 are not unique, then F is said to be Henstock-Kurzweil-Stieltjes-◊-double integrable on $(a,b]_{\mathbb{T}_1}\times(c,d]_{\mathbb{T}_2}$ if it satisfies the following point wise integrability criterion: for every $\varepsilon>0$ there are ◊-gauges δ_1 and δ_2 (or γ_1 and γ_2) defined on $(a,b]_{\mathbb{T}_1}$ and $(c,d]_{\mathbb{T}_2}$ respectively, such that for every $\varepsilon>0$, there are ◊ -gauges δ_1^1 and δ_2^1 (or γ_1^1 and γ_2^1) for $[a,b)_{\mathbb{T}_1}$ and δ_1^2 and δ_2^2 (or γ_1^2 and γ_2^2) for $[c,d)_{\mathbb{T}_2}$ such that

$\parallel S(P^1,F,g)-I_1\parallel_p<\frac{\varepsilon}{2}$ and $\parallel S(P^2,F,g)-I_2\parallel_p<\frac{\varepsilon}{2}$ for all pairs $P^1=P_1^1\times P_2^1$ and $P^2=P_1^2\times P_2^2$ of δ_1-fine (or γ_1)

and for every $\varepsilon>0$ and $i\in\{1,2\}$, there are ◊-gauges δ_1^i and δ_2^i (or γ_1^i and γ_2^i) for $[a,b)_{\mathbb{T}_1}$ and $[c,d)_{\mathbb{T}_2}$ respectively such that

$$\| S(P^i, F, g) - I_i \|_p < \frac{\varepsilon}{2}$$

provided that $P^i = P_1^i \times P_2^i$ is a pair of δ_1^i-fine (or γ_1^i) and δ_2^i-fine (or γ_2^i) partitions of $[a,b)_{\mathbb{T}_1}$ and $[c,d)_{\mathbb{T}_2}$ respectively.

Let $\delta_1 = \min\{\delta_1^1, \delta_1^2\}$ i.e. $(\delta_1)_L = \min\{(\delta_1^1)_L, (\delta_1^2)_L\}$ and $(\delta_1)_R = \min\{(\delta_1^1)_R, (\delta_1^2)_R\}$ and

$\delta_2 = \min\{\delta_2^1, \delta_2^2\}$ i.e. $(\delta_2)_L = \min\{(\delta_2^1)_L, (\delta_2^2)_L\}$ and $(\delta_2)_R = \min\{(\delta_2^1)_R, (\delta_2^2)_R\}$, δ_1 and δ_2 are $\Diamond$-gauges for $(a,b]_{\mathbb{T}_1}$ and $(c,d]_{\mathbb{T}_2}$ respectively, and given a pair $P = P_1 \times P_2$ of δ_1-fine and δ_2-fine partitions of $[a,b)_{\mathbb{T}_1}$ and $[c,d)_{\mathbb{T}_2}$, P_1 is a δ_1^1-fine and δ_1^2-fine partition of $(a,b]_{\mathbb{T}_1}$, P_2 is a δ_2^1-fine and δ_2^2-fine partition of $[c,d)_{\mathbb{T}_2}$, and let $\varepsilon = \frac{\|I_1 - I_2\|_p}{2^{\frac{1}{p}}}$, hence

$$\| I_1 - I_2 \|_p = \left(\| (I_1 - S(P,F,g) + S(P,F,g) - I_2) \|_p\right)$$

$$\leq 2^{\frac{1}{p}}\left(\| S(P,F,g) - I_1 \|_p + \| S(P,F,g) - I_2 \|_p\right)$$

$$< 2^{\frac{1}{p}}\left(\frac{\varepsilon}{2} + \frac{\varepsilon}{2}\right) = 2^{\frac{1}{p}}\varepsilon$$

which is a contradiction. Since for all $\varepsilon > 0$, there are $\Diamond$-gauges δ_1 and δ_2 (or γ_1 and γ_2), then it follows that $I_1 = I_2$. Hence, the Henstock-Kurzweil-Stieltjes-$\Diamond$-double integral of F on $[a,b)_{\mathbb{T}_1} \times [c,d)_{\mathbb{T}_2}$ is unique.

Theorem 3.2. (Bolzano Cauchy Criterion).

A function F is Henstock-Kurzweil-Stieltjes-$\Diamond$-double integrable on $(a,b]_{\mathbb{T}_1} \times (c,d]_{\mathbb{T}_2}$ with respect to monotone increasing functions g_1 and g_2 if and only if for each $\varepsilon > 0$ there exists $\Diamond$-gauges δ_1 and δ_2 (or γ_1 and γ_2) for $[a,b)_{\mathbb{T}_1}$ and $[c,d)_{\mathbb{T}_2}$ respectively, such that $\| S(P^1,F,g) - S(P^2,F,g), F, g) \|_p < \varepsilon$ for all pairs $P^1 = P_1^1 \times P_2^1$ and $P^2 = P_1^2 \times P_2^2$ of δ_1 (or γ_1)-fine partitions of $[a,b)_{\mathbb{T}_1}$ and δ_2 (or γ_2)-fine partitions of $[c,d)_{\mathbb{T}_2}$.

Proof. Suppose F is Henstock-Kurzweil-Stieltjes-$\Diamond$-double integrable on $(a,b]_{\mathbb{T}_1} \times (c,d]_{\mathbb{T}_2}$ with respect to g_1 and g_2 , and let

$$I = \int \int_{\mathcal{R}} F(t,s) \Diamond g_1(t) \Diamond g_2(s).$$

Let $\varepsilon > 0$. There are $\Diamond$-gauges δ_1 and δ_2 for $[a,b)_{\mathbb{T}_1}$ and $[c,d)_{\mathbb{T}_2}$ respectively such that $\| S(P,F,g) - I \|_p < \frac{\varepsilon}{2}$ provided that $P = P_1 \times P_2$ where P_1 is a δ_1 (or γ_1) fine partition of $[a,b)_{\mathbb{T}_1}$ and P_2 is a δ_2 (or γ_2) fine partition of $[c,d)_{\mathbb{T}_2}$. Therefore, if $P = P_1 \times P_2$ and $P = P_1' \times P_2'$ are pairs of δ_1 (or γ_1) fine partition of $[a,b)_{\mathbb{T}_1}$ and P_2 is a δ_2 (or γ_2) fine partition of $[c,d)_{\mathbb{T}_2}$, then

$$\| S(P,F,g) - S(P^1,F,g) \|_p \leq \| S(P,F,g) - I \|_p + \| I - S(P^1,F,g) \|_p$$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Conversely, suppose that for all $\varepsilon > 0$ there are $\diamond$-gauges δ_1 and δ_2 (or γ_1 and γ_2) for $[a,b)_{\mathbb{T}_1}$ and $[c,d)_{\mathbb{T}_2}$ respectively such that $\| S(P^1,F,g) - S(P^2,F,g) \|_p < \varepsilon$ for all pairs $P^1 = P_1^1 \times P_2^1$ and $P^2 = P_1^2 \times P_2^2$ of δ_1 (or γ_1)-fine partitions of $[a,b)_{\mathbb{T}_1}$ and δ_2 (or γ_2)-fine partitions of $[c,d)_{\mathbb{T}_2}$.

Let $n \in \mathbb{N}$. Taking $\varepsilon = \frac{1}{n}$, there are $\diamond$-gauges $\delta_{1,n}$ and $\delta_{2,n}$ (or $\gamma_{1,n}$ and $\gamma_{2,n}$) for $[a,b)_{\mathbb{T}_1}$ and $[c,d)_{\mathbb{T}_2}$ respectively such that $\| S(P',F,g) - S(P^2,F,g) \|_p < \varepsilon$ for all pairs $P^1 = P_1^1 \times P_2^1$ and $P^2 = P_1^2 \times P_2^2$ of $\delta_{1,n}$ (or $\gamma_{1,n}$)-fine partitions of $[a,b)_{\mathbb{T}_1}$ and $\delta_{2,n}$ (or $\gamma_{2,n}$)-fine partitions of $[c,d)_{\mathbb{T}_2}$.

By replacing $\delta_{i,n}$ by $\min\{\delta_{i,1}, \delta_{i,2}, \ldots, \delta_{i,n}\}$ with $i \in \{1,2\}$, we may assume that $\delta_{i,n+1} \leq \delta_{i,n}$. Thus, for all $j > n$ $\delta_{i,j} \leq \delta_{i,n}$ so any pair $P^n = P_1^n \times P_2^n$ of $\delta_{1,n}$ (or $\gamma_{1,n}$)-fine partitions of $[a,b)_{\mathbb{T}_1}$ and $\delta_{2,n}$ (or

$\gamma_{2,n}$)-fine partitions of $[c,d)_{\mathbb{T}_2}$ is also a pair of $\delta_{1,j}$ (or $\gamma_{1,j}$)-fine partitions of $[a,b)_{\mathbb{T}_1}$ and $\delta_{2,j}$ (or $\gamma_{2,j}$)-fine partitions of $[c,d)_{\mathbb{T}_2}$, hence

$$\| S(P^n,F,g) - S(P^j,F,g) \|_p < \frac{1}{j}.$$

This shows that $\{S(P^n,F,g)\}_{n\in\mathbb{N}}$ is a Cauchy sequence.

Let I be the limit of $\{S(P^n,F,g)\}_{n\in\mathbb{N}}$. For all $\varepsilon > 0$, choosing $N > \frac{2}{\varepsilon}$, for ◊-gauges $\delta_{1,N}$ and $\delta_{2,N}$ (or $\gamma_{1,N}$ and $\gamma_{2,N}$) for $[a,b)_{\mathbb{T}_1}$ and $[c,d)_{\mathbb{T}_2}$ respectively. Then there exists N_1 such that for all $n \geq N_1$, we have

$$\| S(P^n,F,g) - I \|_p < \frac{\varepsilon}{2^{\frac{p+1}{p}}}.$$

If there exists N_2 such that $\frac{2^{\frac{p+1}{p}}}{\varepsilon} < N_2$, let $N_0 = \max\{N_1, N_2\}$ and $n \geq N_0$. Let $\delta(s,t) = \delta_{N_1}$ and P be a δ-fine tagged partition of $[a,b)_{\mathbb{T}_1} \times [c,d)_{\mathbb{T}_2}$. Hence,

$$\| S(P,F,g) - S(P^{N_1},F,g) \|_p < \frac{1}{n} \leq \frac{1}{N_0} \leq \frac{1}{N_2} < \frac{\varepsilon}{2^{\frac{p+1}{p}}}.$$

Thus,

$$\| S(P,F,g) - I \|_p \leq \| S(P,F,g) - S(P^{N_1},F,g) + S(P^N,F,g) - I \|_p$$

$$\leq 2^{\frac{1}{p}}\left(\| S(P,F,g) - S(P^{N_1},F,g) \|_p + \| S(P^N,F,g) - I \|_p\right)$$

$$< \frac{1}{N} + \frac{\varepsilon}{2} < 2^{\frac{1}{p}}\left(\frac{\varepsilon}{2^{\frac{p+1}{p}}} + \frac{\varepsilon}{2^{\frac{p+1}{p}}}\right) = \varepsilon.$$

for pair $P = P_1 \times P_2$ such that P_1 is a $\delta_{1,N}$ (or $\gamma_{1,N}$) fine partition of $[a,b)_{\mathbb{T}_1}$ and P_2 is a $\delta_{2,N}$ (or $\gamma_{2,N}$) fine partition of $[c,d)_{\mathbb{T}_2}$.

Theorem 3.3. *Let $F: [a,b]_{\mathbb{T}_1} \times [c,d]_{\mathbb{T}_2} \to l_p$ be a continuous function and $g = g_1 \times g_2$ is of bounded vaiation, then F is Henstock-Kurzweil-Stieltjes-◊-double integrable with respect to g_1 and g_2 on $[a,b)_{\mathbb{T}_1} \times [c,d)_{\mathbb{T}_2}$.*

Proof. Let $\varepsilon > 0$ and g is of bounded variation, there is a positive constant $K > 0$ such that

$$\sum_{i=1}^{n} \sum_{j=1}^{k} |(g_1(t_i) - g_1(t_{i-1}))(g_2(s_j) - g_2(s_{j-1}))| \leq K$$

for any partition P of $[a,b)_{\mathbb{T}_1} \times [c,d)_{\mathbb{T}_2}$. Since F is continuous on $[a,b]_{\mathbb{T}_1} \times [c,d]_{\mathbb{T}_2}$; then, there is a $\lambda > 0$ such that if $|(s,t) - (s_0,t_0)| < \lambda$ for $s,t \in [a,b]_{\mathbb{T}_1} \times [c,d]_{\mathbb{T}_2}$, thus

$$\| F(s,t) - F(s_0,t_0) \|_p < \frac{\varepsilon}{K2^{\frac{p+1}{p}}}.$$

Define $\delta(t,s) > 0$ on $[a,b]_{\mathbb{T}_1} \times [c,d]_{\mathbb{T}_2}$ by $\delta(s,t) = \frac{\lambda}{2}$ for all $s,t \in [a,b]_{\mathbb{T}_1} \times [c,d]_{\mathbb{T}_2}$ and let P_1 is a δ_1 (or γ_1) fine partition of $[a,b]_{\mathbb{T}_1}$ and P_2 is a δ_2 (or γ_2) fine partition of $[c,d]_{\mathbb{T}_2}$. Therefore, if $P = P_1 \times P_2$ and $P = P_1' \times P_2'$ are pairs of δ_1 (or γ_1) fine partition of $[a,b]_{\mathbb{T}_1}$ and P_2 is a δ_2 (or γ_2) fine partition of $[c,d]_{\mathbb{T}_2}$. We have to establish that

$$\| S(P_1,F,g) - S(P_2,F,g) \|_p < \varepsilon.$$

Without loss of generality, and that P_1 is a refinement of P_2. Thus, we establish the following cases:

(i). Consider the case where the tagged partitions P_1, P_2 and P_1', P_2' are different in their tags. That is, we have $P_1 = \{t_0, t_1, \dots, t_n\} \subset [a,b]_{\mathbb{T}_1}$ with tag points $\xi_i \in [t_{i-1}, t_i]_{\mathbb{T}_1}$ for $i = 1, \dots, n$ is a δ_1-fine (or γ_1) and $P_2 = \{s_0, s_1, \dots, s_k\} \subset [c,d]_{\mathbb{T}_2}$ with tag points $\zeta_j \in [s_{j-1}, s_j]_{\mathbb{T}_2}$, $j = 1,2,\dots,k$ is a δ_2-fine (or γ_2) are δ-fine (or γ) partitions of $[a,b)_{\mathbb{T}_1}$ and $[c,d)_{\mathbb{T}_2}$ respectively and also, P_1', P_2' with the tags $\xi_i' \in [t_{i-1}, t_i]_{\mathbb{T}_1}$ for $i = 1, \dots, n$ and $\zeta_j' \in [s_{j-1}, s_j]_{\mathbb{T}_2}$, $j = 1,2,\dots,k$ where $(\xi_i, \zeta_j) \neq (\xi_i', \zeta_j')$ for each i, j.

Let $\lozenge\, g_{1_i} \lozenge\, g_{2_j} = (g_1(t_i) - g_1(t_{i-1}))(g_2(s_j) - g_2(s_{j-1}))$. Hence,

$$\| S(P_1, F, g_1, g_2) - S(P_2, F, g) \|_p = \| \sum_{i=1}^n \sum_{j=1}^k F(\xi_i, \zeta_j) \lozenge\, g_{1_i} \lozenge\, g_{2_j} - \sum_{i=1}^n \sum_{j=1}^k F(\xi_i', \zeta_j') \lozenge\, g_{1_i} \lozenge\, g_{2_j} \|_p$$

$$\leq 2^{\frac{p+1}{p}} \sum_{i=1}^n \sum_{j=1}^k \| F(\xi_i, \zeta_j) \lozenge\, g_{1_i} \lozenge\, g_{2_j} - F(\xi_i', \zeta_j') \lozenge\, g_{1_i} \lozenge\, g_{2_j} \|_p$$

$$< 2^{\frac{p+1}{p}} \sum_{i=1}^n \sum_{j=1}^k \left[| \lozenge\, g_{1_i} \lozenge\, g_{2_j} | \left(\frac{\varepsilon}{K2^{\frac{p+1}{p}}} \right) \right]$$

$$\leq \left(\frac{\varepsilon}{K} \right) (K) = \varepsilon.$$

(ii). Suppose that P_2 is a refinement of P_1 by adding one point, say s_j' for $s_{j-1} < s_j' < s_j$ with $P_2 = \{(s_j', [s_{j-1}, s_j]_{\mathbb{T}_2}, j = 1,2,\dots,k\} \subset [c,d]_{\mathbb{T}_2}$ with tag points $\zeta_j' \in [s_{j-1}, s_j]_{\mathbb{T}_2}$, $j = 1,2,\dots,k$ is a δ_2-fine (or γ_2) are δ-fine (or γ) partitions of $[a,b)_{\mathbb{T}_1}$ and $[c,d)_{\mathbb{T}_2}$

Hence $\| S(P_2, F, g_1, g_2) - S(P_1, F, g_1, g_2) \|_p$

$$= \| F(\xi_{i-1}, \zeta_{j-1}')[(g_1(t_i) - g_1(t_{i-1}))(g_2(s_j') - g_2(s_{j-1}))]$$

$$+ F(\xi_i, \zeta_j)[(g_1(t_i) - g_1(t_{i-1}))(g_2(s_j) - g_2(s_{j-1}'))] - F(\xi_i, \zeta_j)[(g_1(t_i) - g_1(t_{i-1}))(g_2(s_j) - g_2(s_{j-1}))] \|_p$$

$$= \| [(g_1(t_i) - g_1(t_{i-1}))(g_2(s_j') - g_2(s_{j-1}))] \| \| F(\xi_{i-1}, \zeta_{j-1}') - F(\xi_i, \zeta_j) \|_p$$

$$< K \left(\frac{\varepsilon}{K2^{\frac{P+1}{p}}} \right) \leq \varepsilon.$$

This completes the proof.

We present the following algebraic properties of the l_p-valued Henstock-Kurzweil-Stieltjes-$\lozenge$-double integral on time scales.

Theorem 3.4. *Let $F, H: [a,b)_{\mathbb{T}_1} \times [c,d)_{\mathbb{T}_2} \to l_p$ be an l_p-valued function with g_1 and g_2 be monotone increasing functions respectively on $[a,b]_{\mathbb{T}_1}$ and $[c,d]_{\mathbb{T}_2}$. If F is Henstock-Kurzweil-Stieltjes-$\lozenge$-double integrable with respect to g_1 and g_2 on $\mathcal{R} = [a,b)_{\mathbb{T}_1} \times [c,d)_{\mathbb{T}_2}$. Then,*

i. $\int \int_{\mathcal{R}} \lambda F(t,s) \lozenge\, g_1(t) \lozenge\, g_2(s) = \lambda \int \int_{\mathcal{R}} F(t,s) \lozenge\, g_1(t) \lozenge\, g_2(s)$, λ isaconstant;

ii. $\int \int_{\mathcal{R}} (F(t,s) + H(t,s)) \lozenge\, g_1(t) \lozenge\, g_2(s) = \int \int_{\mathcal{R}} F(t,s) \lozenge\, g_1(t) \lozenge\, g_2(s) + \int \int_{\mathcal{R}} H(t,s) \lozenge\, g_1(t) \lozenge\, g_2(s)$.

Proof. (i) Let F be Henstock-Kurzweil-Stieltjes-⋄-double integrable with respect to monotone increasing functions g_1 and g_2 on $[a,b)_{\mathbb{T}_1} \times [c,d)_{\mathbb{T}_2}$. Let λ be a constant. The case $\lambda = 0$ is trivial. Suppose $\lambda \neq 0$. Since F Henstock-Kurzweil-Stieltjes-$\lozenge$-double integrable, there is a function $\delta(t,s) > 0$ on $[a,b)_{\mathbb{T}_1} \times [c,d)_{\mathbb{T}_2}$ such that

$$\| S(P, F, g_1, g_2) - \int \int_{\mathcal{R}} F(t,s) \lozenge\, g_1(t) \lozenge\, g_2(s) \|_p < \frac{\varepsilon}{|\lambda|}$$

whenever P is a δ-fine tagged partition of $[a,b)_{\mathbb{T}_1} \times [c,d)_{\mathbb{T}_2}$. Then,

$$\| S(P,\lambda F,g_1,g_2) - \lambda \int\int_{\mathcal{R}} F(t,s) \lozenge g_1(t) \lozenge g_2(s) \|_p = \| \lambda S(P,F,g_1,g_2) - \lambda \int\int_{\mathcal{R}} F(t,s) \lozenge g_1(t) \lozenge g_2(s) \|_p$$

$$< |\lambda| \left(\frac{\varepsilon}{|\lambda|}\right) = \varepsilon.$$

(ii) Let $\varepsilon > 0$.

Suppose $\int\int_{\mathcal{R}} F(t,s) \lozenge g_1(t) \lozenge g_2(s) = I_1$ and $\int\int_{\mathcal{R}} H(t,s) \lozenge g_1(t) \lozenge g_2(s) = I_2$. There exists a positive function δ_1 on $[a,b)_{\mathbb{T}_1} \times [c,d)_{\mathbb{T}_2}$ when $P_1 = \{t_0, t_1, \ldots, t_n\} \subset [a,b]_{\mathbb{T}_1}$ with tag points $\xi_i \in [t_{i-1}, t_i]_{\mathbb{T}_1}$ for $i = 1, \ldots, n$ and a δ_1-fine (or γ_1) such that

$$\| S(P_1,F,g_1,g_2) - I_1 \|_p < \frac{\varepsilon}{2\left(2^{\frac{1}{p}}\right)}.$$

Also, there is a function $\delta_2 > 0$ when $P_2 = \{s_0, s_1, \ldots, s_k\} \subset [c,d]_{\mathbb{T}_2}$ with tag points $\zeta_j \in [s_{j-1}, s_j]_{\mathbb{T}_2}$, $j = 1,2,\ldots,k$ is a δ_2-fine (or γ_2) such that

$$\| S(P_1,F,g_1,g_2) - I_1 \|_p < \frac{\varepsilon}{2\left(2^{\frac{1}{p}}\right)}.$$

Define a function $\delta(t,s) > 0$ on $[a,b)_{\mathbb{T}_1} \times [c,d)_{\mathbb{T}_2}$ by $\delta(t,s) = \min\{\delta_1, \delta_2\}$ for all $(t,s) \in [a,b)_{\mathbb{T}_1} \times [c,d)_{\mathbb{T}_2}$. Let P be a δ-fine tagged partition of $[a,b)_{\mathbb{T}_1} \times [c,d)_{\mathbb{T}_2}$. Then

$$\| S(P,F+H,g_1,g_2) - (I_1 + I_2) \|_p = \| S(P,F,g_1,g_2) + S(P,H,g_1,g_2) - (I_1 + I_2) \|_p$$

$$\leq 2^{\frac{1}{p}} (\| S(P,F,g_1,g_2) - I_1) \|_p + \| S(P,H,g_1,g_2) - I_2) \|_p)$$

$$< 2^{\frac{1}{p}} \left(\frac{\varepsilon}{2(2^{\frac{1}{p}})} + \frac{\varepsilon}{2(2^{\frac{1}{p}})} \right) = \varepsilon.$$

4. Conclusion

In this article, we defined Henstock-Kurzweil-Stieltjes-◊-double integrals for l_p-valued functions on time scales. Some of the basic properties of this integral with values ranging in an l_p-space, with $0 < p < 1$ are discussed on time scales.

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