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Article

Direct Solutions of Second-Order Ordinary Differential Equations Using Linear Multi-Step Method

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Abstract

The analytical solutions to second-order ODEs often prove elusive, necessitating the development of numerical methods for their approximation. However, solving second-order ordinary differential equations (ODEs) directly with linear multi-step methods abridges the transformation process of changing the equations to a system of first order equations before solving. The aim of this study is to develop and optimize a direct solution methodology for second-order ordinary differential equations (ODEs) by comparing the step length of linear multi-step methods. Having developed a new class of continuous linear multi step methods, initial value problems of general second order ordinary differential equations was solved via collocation and interpolation technique on the Chebyshev polynomial equations. The three-step method together with the four-step method that was developed were analyzed based on the properties of linear multistep methods and were found to be zero -stable, consistent and convergent with good region of absolute stability. The new class of linear multi step methods has the advantage of evaluation of functions at different step lengths. These proposed methods were implemented on second order ordinary differential initial value problems. The performance of the new methods was tested and compared to the exact solution of two worked examples. Hence, the computed results together with its associated errors shows that the equation converges faster as the step length reduces.

Keywords: Chebyshev polynomial, Linear multistep methods, Second-order ordinary differential equations.

1. Introduction

In the realm of mathematics, the study of ordinary differential equations (ODEs) holds significant importance due to its widespread applications in various scientific and engineering disciplines. Second-order ordinary differential equations, in particular, represent a fundamental class of equations that arise in diverse fields such as physics, biology, economics, and engineering.

The analytical solutions to second-order ODEs often prove elusive, necessitating the development of numerical methods for their approximation. Among these methods, linear multi-step methods stand out as powerful tools for approximating the solutions of ODEs with high accuracy and efficiency. This project focuses on exploring and implementing direct solutions of second-order ordinary differential equations of the form:

$$y^{(m)}(x) = f\left(x, y, y^{'}, ...y^{(m-1)}\right) \qquad \textbf{(1)}$$

using linear multi-step methods. Second-order ODEs, governing systems with two degrees of freedom, play a prevalent role in predictive modeling, especially in mechanical systems, electrical circuits, and chemical reactions. Unlike traditional methods that rely on iterative processes, direct methods offer a systematic *app*roach to obtaining approximate solutions by advancing the solution directly from one point to another, without intermediate steps. This characteristic makes them particularly appealing for practical applications where computational efficiency is crucial.

The primary objective of this study is to investigate the theoretical foundations and computational aspects of linear multi-step methods for solving second-order ODEs. By analyzing the convergence properties, stability criteria, and computational complexity of these methods, we aim to provide insights into their suitability for different types of differential equations and boundary value problems.

In summary, this research endeavor represents a systematic exploration of direct solutions to second-order ordinary differential equations using linear multi-step methods. Through theoretical analysis, computational experimentation, and software development, we aim to advance our understanding and utilization of these methods, thereby contributing to the broader landscape of numerical mathematics and scientific computing.

2. Literature Review

In the study conducted by [1], the focus was on exploring the numerical solutions of initial value problems pertaining to general second-order ordinary differential equations. The research involved the development of a novel class of continuous implicit hybrid one-step methods, leveraging the collocation and interpolation technique applied to the power series approximate solution. These methods were designed to overcome the Dahlquist zero stability barrier and enhance the consistency order by incorporating off-step points. [1] work highlighted the advantageous features of the new methods, such as the flexibility in adjusting step length and the ease of function evaluation at off-step points. The implementation of the main method using Block method ensured that each discrete method derived from the simultaneous solution of the block maintained the same level of accuracy. As a result, the new class of one-step methods exhibited high accuracy, low error constants, extensive intervals of absolute stability, zero stability, and convergence. The effectiveness of the methods was validated through sample examples involving linear, nonlinear, and stiff problems, where computed results were compared with exact solutions and errors of existing methods in terms of step number and order of accuracy. The study employed efficient computer codes for computation and analysis [1].

In their paper, [3] wrote a paper on introducing new method for directly solving a specific type of mathematical problem involving second-order equations. They developed a technique called a linear multistep method, which has a fifth order of accuracy and can start solving the problem without using information from other methods. The method is created by using mathematical tools like interpolation and collocation. These tools help establish a system of equations that the authors solve to find the necessary coefficients. The resulting method is then applied to solve problems without needing additional information or initial guesses. The authors provide numerical examples to demonstrate the effectiveness of their approach.

Jator[3] proposed a self-starting linear multistep method of order 5 for directly solving the general second-order initial value problem (IVP). The method is formulated through interpolation and collocation techniques applied to the assumed approximate solution and its second derivative at specific grid points. Specifically, the interpolation and collocation procedures yield a system of (r + s) equations involving (r + s) unknown coefficients, which are subsequently determined using the matrix inversion approach. This self-starting linear multistep method offers a direct solution to the general second-order IVP without requiring predictors or starting values from other methods. The effectiveness of the method is demonstrated through numerical examples provided in the paper.

Various researchers including[6], [7], [4] and [2] have discussed the solution of initial value problems in second-order ODEs of the form $y' = f\left(x, y, y', \ldots, \right)$ Another method explored is the hybrid method, combining characteristics of continuous linear multi-step methods and Runge-Kutta methods.

This approach utilizes data at points other than the step point. The hybrid method aims to overcome the drawbacks of previous approaches.

To address these limitations, the Block method was introduced. This method simultaneously generates approximations at different grid points within the integration interval and is less computationally expensive than linear multistep methods or Runge-Kutta methods.

3. Specification of the Methods

3.1. The Derivation of three Step Method

In this section, we use Chebyshev polynomial as an approximate solution to be of the form:

$$y(x) = \sum_{i=0}^{p+q-1} \alpha_i T_i(x) \qquad \textbf{(2)}$$

where p represents the number of collocation points and q is the number of interpolating points.

The second derivative of equation **(2)** gives:

$$y''(x) = \sum_{i=0}^{p+q-1} \alpha_i T''_i(x) = f\left(x, y, y'\right) \qquad \textbf{(3)}$$

Equations **(2)** and **(3)** are interpolated and collocated at the points P = 0, and1 and q = 0, 1,2 and 3 to get a system of equations of the form:

AX = B **(4)**

where $X = \left(a_0, a_1, a_2, a_3, a_4, a_5\right)^T$, $Y = \left(y_0, y_1, f_0, f_1, f_2, f_3\right)^T$ and

$$A := \begin{bmatrix} 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 4 & -24 & 80 & -200 \\ 0 & 0 & 4 & 0 & -16 & 0 \\ 0 & 0 & 4 & 24 & 80 & 200 \\ 0 & 0 & 4 & 48 & 368 & 2320 \end{bmatrix} \qquad \textbf{(5)}$$

By simplifying some notations in equation **(4)** and solving for a_j'j = 0,1,2,3,4 and 5 and substituting the value of a_j's into equation **(2)** gives a continuous implicit multistep method of the form:

$$\alpha_0 y_0 + \alpha_1 y_1 + \sum_{j=o}^{3} \beta_j f_j \qquad \textbf{(6)}$$

Where j = 0, 1 ,2 and 3. α_j, β_j represent continuous coefficients, y_j represents numerical solution at point x_j and $f_j = f\left(x_j, y_j, y'_j\right)$.

Using the transformation, the coefficients α_j and β_j are given as:

$$\alpha_0(Z) = 1 - Z$$
$$\alpha_1(Z) = Z$$
$$\beta_0(Z) = -\frac{97}{360}Z + \frac{1}{2}Z^2 - \frac{11}{36}Z^3 + \frac{1}{12}Z^4 - \frac{1}{120}Z^5$$
$$\beta_1(Z) = -\frac{19}{60}Z + \frac{1}{2}Z^3 - \frac{5}{24}Z^4 + \frac{1}{40}Z^5$$
$$\beta_2(Z) = \frac{13}{120}Z - \frac{1}{4}Z^3 + \frac{1}{6}Z^4 - \frac{1}{40}Z^5$$
$$\beta_3(Z) = -\frac{1}{45}Z + \frac{1}{18}Z^3 + \frac{1}{120}Z^5 - \frac{1}{24}Z^4 \tag{7}$$

Evaluating equation **(6)** together with **(7)** at p = 1 and q = 3 we get

$$y_{n+1}(x) = y_1 \tag{8}$$
$$y_{n+3}(x) = -2y_0 + 3y_1 + h\left(\frac{1}{6}F_0 + \frac{7}{4}F_1 + F_2 + \frac{1}{12}F_3\right) \tag{9}$$

The first derivatives of the equations **(7)** give:

$$\alpha'_0(Z) = -1$$
$$\alpha'_1(Z) = 1$$
$$\beta'_0(Z) = -\frac{97}{360} + Z - \frac{11}{12}Z^2 + \frac{1}{3}Z^3 - \frac{1}{24}Z^4$$
$$\beta'_1(Z) = -\frac{19}{60} + \frac{3}{2}Z^2 - \frac{5}{6}Z^3 + \frac{1}{8}Z^4$$
$$\beta'_2(Z) = \frac{13}{120} - \frac{3}{4}Z^2 + \frac{2}{3}Z^3 - \frac{1}{8}Z^4$$
$$\beta'_3(Z) = -\frac{1}{45} + \frac{1}{6}Z^2 + \frac{1}{24}Z^4 - \frac{1}{6}Z^3 \tag{10}$$

By evaluating the first derivative equation (6) together with **(7)** at points p = 0, 1,2 and 3, we obtain:

$$y'_n(x) = h\left(-\frac{97}{360}F_0 - \frac{19}{60}F_1 + \frac{13}{120}F_2 - \frac{1}{45}F_3\right) \tag{11}$$

$$y'_{n+1}(x) = h\left(\frac{19}{180}F_0 + \frac{19}{40}F_1 - \frac{1}{10}F_2 + \frac{7}{360}F_3\right) \tag{12}$$

$$y'_{n+2}(x) = h\left(\frac{23}{360}F_0 + \frac{61}{60}F_1 + \frac{53}{120}F_2 - \frac{1}{45}F_3\right) \tag{13}$$

$$y'_{n+3}(x) = h\left(\frac{19}{180}F_0 + \frac{97}{120}F_1 + \frac{37}{30}F_2 + \frac{127}{360}F_3\right) \tag{14}$$

In order to derive the block methods and to test for the convergence, we combine equations **(10)** and **(11)** using their coefficients in the block form. After normalizing and substituting where necessary, we rewrite the equation explicitly as:

$$y_{n+1} = y_n + y'_n + h\left(\frac{97}{360}f_n + \frac{19}{60}f_{n+1} - \frac{13}{120}f_{n+2} + \frac{1}{45}f_{n+3}\right) \quad \textbf{(15)}$$

$$y_{n+2} = y_n + 2y'_n + h\left(\frac{28}{45}f_n + \frac{22}{15}f_{n+1} - \frac{2}{15}f_{n+2} + \frac{2}{45}f_{n+3}\right) \quad \textbf{(16)}$$

$$y_{n+3} = y_n + 3y'_n + h\left(\frac{39}{40}f_n + \frac{27}{10}f_{n+1} + \frac{27}{40}f_{n+2} + \frac{3}{20}f_{n+3}\right) \quad \textbf{(17)}$$

$$y'_{n+1} = y_n + h\left(\frac{3}{8}f_n + \frac{19}{24}f_{n+1} - \frac{5}{24}f_{n+2} + \frac{1}{24}f_{n+3}\right) \quad \textbf{(18)}$$

$$y'_{n+2} = y_n + h\left(\frac{1}{3}f_n + \frac{4}{3}f_{n+1} + \frac{1}{3}f_{n+2}\right) \quad \textbf{(19)}$$

$$y'_{n+3} = y_n + h\left(\frac{3}{8}f_n + \frac{9}{8}f_{n+1} + \frac{9}{8}f_{n+2} + \frac{3}{8}f_{n+3}\right) \quad \textbf{(20)}$$

3.2. The Derivation of the Four-Step Method

Consider the step number k = 0,1,2,3, and 4 as the off -step points. Following similar procedure, the method obtained is:

$$y_{n+1} = y_n + y'_n + h^2\left(\frac{1231}{5040}f_n + \frac{3}{8}f_{n+1} - \frac{47}{240}f_{n+2} + \frac{29}{360}f_{n+3} - \frac{7}{480}f_{n+4}\right): \quad \textbf{(21)}$$

$$y_{n+2} = y_n + 2y'_n + h^2\left(\frac{71}{126}f_n + \frac{8}{5}f_{n+1} - \frac{1}{3}f_{n+2} + \frac{8}{45}f_{n+3} - \frac{1}{30}f_{n+4}\right): \quad \textbf{(22)}$$

$$y_{n+3} = y_n + 3y'_n + h^2\left(\frac{123}{140}f_n + \frac{117}{40}f_{n+1} + \frac{27}{80}f_{n+2} + \frac{3}{8}f_{n+3} - \frac{9}{160}f_{n+4}\right): \quad \textbf{(23)}$$

$$y_{n+4} = y_n + 4y'_n + h^2\left(\frac{376}{315}f_n + \frac{64}{15}f_{n+1} + \frac{16}{15}f_{n+2} + \frac{64}{45}f_{n+3}\right): \quad \textbf{(24)}$$

$$y'_{n+1} = y_n + h\left(\frac{95}{288}f_n + \frac{323}{360}f_{n+1} - \frac{11}{30}f_{n+2} + \frac{53}{360}f_{n+3} - \frac{19}{720}f_{n+4}\right): \quad \textbf{(25)}$$

$$y'_{n+2} = y_n + h\left(\frac{14}{45}f_n + \frac{62}{45}f_{n+1} + \frac{4}{15}f_{n+2} + \frac{2}{45}f_{n+3} - \frac{1}{90}f_{n+4}\right):$$

(26)

$$y'_{n+3} = y_n + h\left(\frac{51}{160}f_n + \frac{51}{40}f_{n+1} + \frac{9}{10}f_{n+2} + \frac{21}{40}f_{n+3} - \frac{3}{80}f_{n+4}\right):$$

(27)

$$y'_{n+4} = y_n + h\left(\frac{14}{45}f_n + \frac{64}{45}f_{n+1} + \frac{8}{15}f_{n+2} + \frac{64}{45}f_{n+3} + \frac{14}{45}f_{n+4}\right):$$

(28)

4. Analysis of the Methods

The methods have some basic properties which establish their validity. The properties: order error constant, consistency and zero stability reveal the nature of convergence of the methods.

4.1. Order and Error Constant

Following (Henrici, 1962) the approach adopted in (Fatunla, 1991,1994) and (Lambert, 1973), we define the truncation error associated with equation **(20)** by the difference:

$$L(y(x)) = \sum_{j=0}\left[\alpha_j y(x_n + jh) - h^2 \beta_j y''(x_n + jh)\right\}$$ **(29)**

where y(x) is an arbitrary test function continuously differential on [a, b]. Expanding **(29)** in Taylor series about the point x, we obtain the expression:

$$L(y(x)) = c_0 y(x) + c_1 h y'(x) + c_2 h^2 y''(x) + c_3 h^3 y'(x) + \ldots + c_{p+3} 3 h^{p+3} y^{p+3}(x)$$

(30)

where the c_0, c_1, c_2, c_p are obtained as:

$$c_0 = \sum_{j=0}^{k} \alpha_j$$ **(31)**

$$c_1 = \sum_{j=1}^{k} j\alpha_j$$ **(32)**

$$c_2 = \frac{1}{2}\sum_{j=2}^{k} j^2 \alpha_j$$ **(33)**

$$c_q = \frac{1}{q!}\left[\sum_{j=1}^{k} j^q \alpha_j - q(q-1)(q-2)\sum_{j=1}^{k} j^q \beta_j j^{q-3}\right]$$ **(34)**

According to Lambert [4], the equations above are of order p if $c_0 = c_1 = c_2 = \ldots c_p = c_{p+1} = 0$ and $c_{p+2} \neq 0$.

The $c_{p+2} \neq 0$ is called the error constant and $c_{p+2} h^{p+2} y^{p+2}(x_n)$ is the principal local truncation error at the point (x_n) .

Using Taylor series expansion on the equations above, we get the order the new proposed block methods respectively as (3,3,5,4,3,3) and (3,3,3,3,3,3) with error constants as:

$$\left(-\frac{1}{3}, 2, \frac{81}{8}, \frac{1}{2}, 2, \frac{3}{2}\right)$$

and

$$\left(\frac{5147}{10080}, \frac{638}{315}, \frac{9}{224}, -\frac{1664}{315}, \frac{3}{160}, \frac{181}{90}, \frac{243}{160}, -\frac{4}{3}\right).$$

Consequently, we find the Zero Stability, consistency and Convergence.

4.2. Convergence

The convergence of the proposed method is considered in the light of the basic properties discussed in the fundamental theorem of Dahlquist (Henrici, 1962) for linear multistep method.

5. Numerical Example

In this section, some numerical examples of second order ordinary differential equations are solved. The methods are implemented directly without using any starting value and with use of Maple. The table below shows some notations that were used to present the numerical and graphical results obtained for some test problems by application of the proposed schemes.

X	Point of evaluation
K	Step Number
3SL	3 Step length (Numerical Solution)
4SL	4 Step Length (Numerical Solution)

Problem 1

Consider a Linear non-homogeneous test problem

$$y''(x) = y'(x) + x e^{3x}, \; y'(0) = -\frac{5}{32}, \; y(0) = -\frac{3}{32}, h = 0.0025$$

Exact solution: $y(x) = \frac{5}{32} - \frac{5}{32} e^{x} - \frac{3}{32}$

Problem 2

Consider a special oscillatory test problem

$$y''(x) + y'(x) = 0.01\cos(x), y(0) = 1, y'(0) = -1, h = 0.01$$

Exact solution: $y(x) = -0.995e^{-x} + 1.000 - 0.005\cos(x) - 0.005\sin(x)$

Tabular Presentations

Here we present numerical and error results for 3SL and 4SL in the tables below:

Table 1a: Numerical Results for Problem 1

X	Exact Solution	Numerical Solution (3 step length)	Numerical Solution (4 step length)
0.0000	-0.093750000000000000	-0.949244516083860535e-1	-0.953170146283212283e-1
0.0025	-0.093735734819276141	-0.949243796965873289e-1	-0.953168869432698883e-1
0.0050	-0.093721468728007991	-0.949243067060864698e-1	-0.953167573453311414e-1
0.0075	-0.093707201726195109	-0.949242326247176250e-1	-0.953166258129401949e-1
0.0100	-0.093692933813836512	-0.949241574401929032e-1	-0.953164923243159784e-1
0.0125	-0.093678664990931338	-0.949240811401012257e-1	-0.953163568574591099e-1
0.0150	-0.093664395257478630	-0.949240037119071676e-1	-0.953162193901498431e-1
0.0175	-0.093650124613477412	-0.949239251429497888e-1	-0.953160798999459964e-1
0.0200	-0.093635853058928638	-0.949238454204414554e-1	-0.953159383641808630e-1
0.0225	-0.093621580593831425	-0.949237645314666487e-1	-0.953157947599611022e-1
0.0250	-0.093607307218185716	-0.949236824629807654e-1	-0.953156490641646119e-1

Table 1b: Errors of Methods for Problem 1

X	Error 1	Error 2
0.0000	2.055484168639465e-12	1.931701462832123e-2
0.0025	1.265635266087719e-10	1.931680408076849e-2
0.0050	2.401197300173001e-10	1.931659615847920e-2
0.0075	3.512568387013116e-10	1.931639104084811e-2
0.0100	4.616934319855932e-10	1.931618880589323e-2
0.0125	5.723759034202312e-10	1.931598953104961e-2
0.0150	6.841444718693045e-10	1.931579329563200e-2
0.0175	7.978918614724840e-10	1.931559017438748e-2
0.0200	9.144017714170382e-10	1.931538024580791e-2
0.0225	1.034758342042708e-09	1.931516358675785e-2
0.0250	1.158511290901019e-09	1.931494027899810e-2

Table 2a: Numerical Results for Problem 2

X	Exact Solution	Numerical Solution (3 step length∣)	Numerical Solution (4 step length∣)
0.00	1.000000000000000000	0.970449988548573513	0.970445996108059132
0.01	0.997499999687498055	0.970449987879207812	0.970445995440669870
0.02	0.995000000249995833	0.970449986764412759	0.970445994331803936
0.03	0.992500000937491973	0.970449985204299831	0.970445992781572217
0.04	0.990000001249991073	0.970449983199025039	0.970445990790129733
0.05	0.987499999374989859	0.970449980748788910	0.970445988357675628
0.06	0.985000000812491544	0.970449977853836463	0.970445985484453144
0.07	0.982500000499991318	0.970449974514457193	0.970445982170749602
0.08	0.980000001062489196	0.970449970730985034	0.970445978416896369
0.09	0.977499999187487983	0.970449966503798331	0.970445974223268828
0.10	0.974999999749990761	0.970449961833319798	0.970445969590286337

Table 2b: Errors of Methods for Problem 2

X	Error 1	Error 2
0.00	0.029550011451426487	0.029554003891940868
0.01	0.027050011808290243	0.027054004246828185
0.02	0.024550013485583074	0.024554005918191897
0.03	0.022050015733192142	0.022054008155919756
0.04	0.019550018050966034	0.019554010459861340
0.05	0.017050019626201949	0.017054012017314231
0.06	0.014550022958655081	0.014554015328038400
0.07	0.012050026985534125	0.012054019329848716
0.08	0.009550030331504162	0.009554022645572827
0.09	0.007050032683689652	0.007054025964219155
0.10	0.004550037916670963	0.004554030159703424

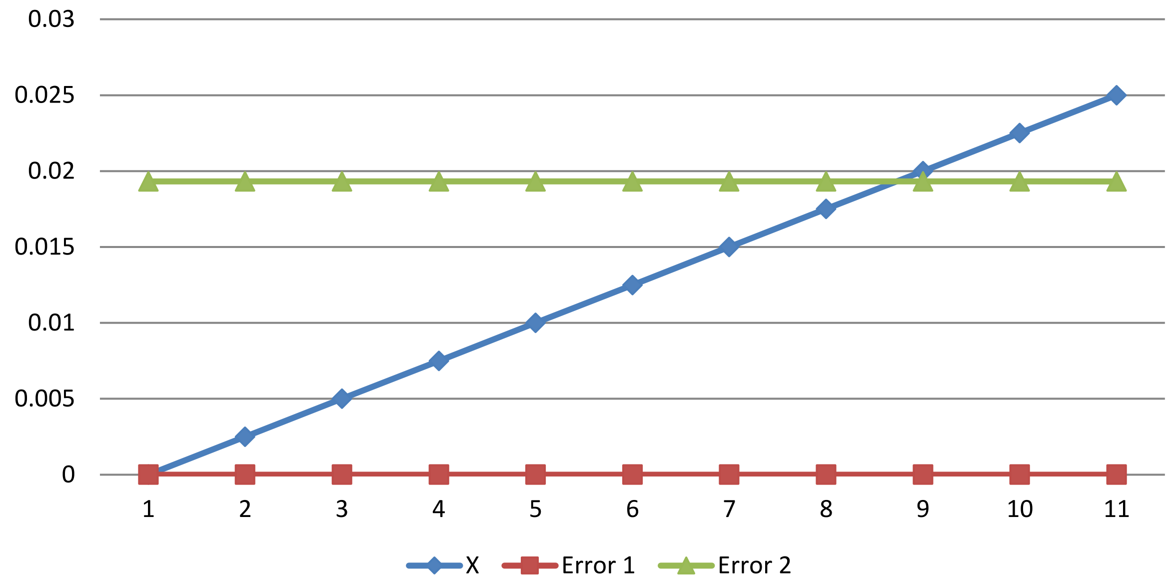

Figure 1: Graphical Presentation of Error of Methods for Problem 1. This graph shows that the errors of the proposed methods are less in magnitude compared to that of the exact and the higher step length (Error 2) is relatively closer in its values to the exact solution as the graph shows that it is much closer to the horizontal axis.

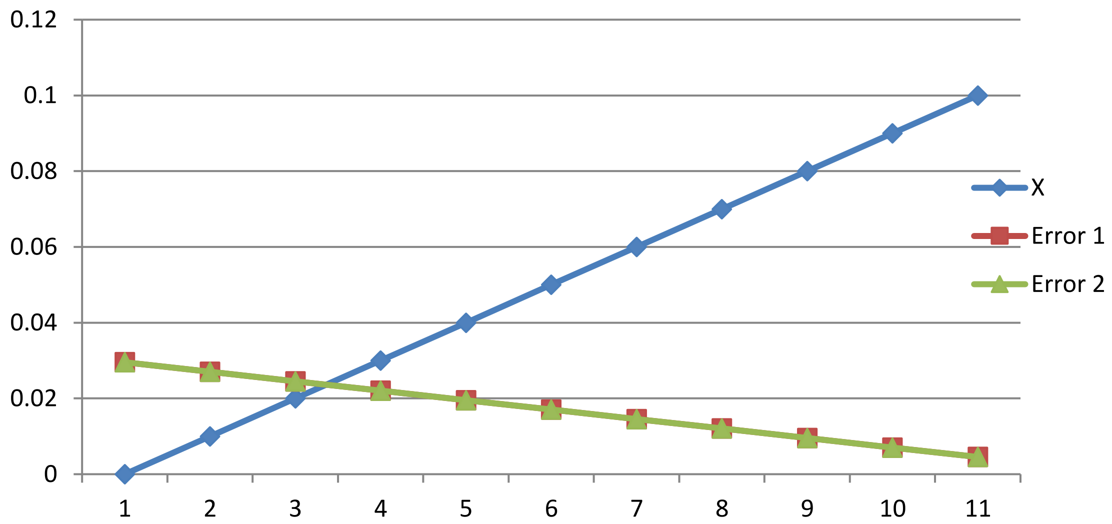

Figure 2: Graphical Presentation of Error of Methods for Problem 2. This graph shows that the errors of the proposed methods are closely similar to that of the exact and the higher step length (Error 2) is relatively closer in its values to the exact solution as the graph shows that it is much closer to the horizontal axis

6. Discussion of Results

In this work, we developed 3rd and 4th order methods for solving initial value problems directly without reducing the equations to a first order differential equation. The method that was developed was tested by using it to solve numerical examples which are linear and non linear initial value problems of second order ordinary differential equation. The table of results of our method is shown above comparing the proposed method with the exact solution to observe and infer that the results are better as the step length reduces.

7. Conclusion

In this work, two numerical schemes were developed in which natural numbers were used as the Step-lengths for second order ordinary differential equations. The resulting methods are consistent and zero stable, therefore it convergences. The methods have good region of absolute stability. The table and graphical presentation of numerical results of the problem show that the method is effective and accurate compared with the exact solution and the results get better as the step length reduces.

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Conflict of Interest
The author declared no conflict of interest in the manuscript.

Authors' Declaration
The author(s) hereby declare that the work presented in this article is original and that any liability for claims relating to the content of this article will be borne by them.

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