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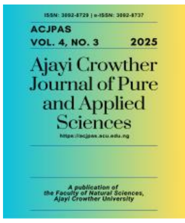

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Article

Hyers-Ulam Stability Criteria of Nonlinear Second Order Differential Equations with Nonlinear Damping Terms

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Abstract

This work examines the Hyers-Ulam stability criteria for second order nonlinear differential equations of nonlinear damping terms. A key component of our findings is the conversion of all second order nonlinear differential equations considered using criteria developed into integral inequalities for Gronwall-Bellman-Bihari integral inequality to be easily applied. The results obtained extend all the known findings on the stability for second order nonlinear differential equations in the literature. We provide examples to support the validity of our theorems.

Keywords: Gronwall-Bellman-Bihari type inequality, Hyers-Ulam stability, Integral inequality, Nonlinear damping term.

1. Introduction

The stability problem of functional equation started with the question concerning stability of group homomorphism proposed by Ulam[1] in 1940 during a talk before a Mathematical Colloquium at University of Wincosin, Maidison. In 1941, Hyers[2] gave a solution of Ulam's problem for the case of appropriate additive mappings in the context of Banach spaces. Hyers [2] demonstrated the following result on Banach spaces, which is now known as the Hyers-Ulam stability (HUS) theorem. "Let $\mathbf{L_1}$, $\mathbf{L_2}$ be real Banach spaces and $\epsilon > 0$. Then for every Mapping $\eta: \mathbf{L_1} \to \mathbf{L_2}$ satisfying

$$||\eta(\xi + \varepsilon) - \eta(\xi) - \eta(\varepsilon)|| \leq \epsilon,$$

$\forall\, \xi, \varepsilon \in \mathbf{L_1}$ there $\exists$ a unique additive mapping $\rho: \mathbf{L_1} \to \mathbf{L_2}$ with the property

$$||\eta(\xi) - \rho(\xi)|| \leq \epsilon,$$

$\forall\, \xi \in \mathbf{L_1}$."

Numerous studies have been published since Hyers' result, expanding and generalizing Ulam's and Hyers' difficulties in different ways (see [3, 4, 5, 6, 7, 8, 9, 10]). The following second order nonlinear differential equations (SONDE) will be examined for HUS in this work, with nonlinear damping terms provided as:

$$(r(t)\psi(u(t))(u'(t))^{\alpha})' + p(t)(u'(t))^{\alpha} + q(t)f(u(t)) = \Re(t, u(t), u'(t)), \qquad (1.1).$$

$$(r(t)\psi(u(t))(u'(t))' + p(t)\Bbbk(t,u(t),u'(t))(u'(t)) + q(t)f(u(t)) \\ = \Re(t,u(t),u'(t)), \tag{1.2}$$

$$(r(t)(u'(t))' + p(t)(u'(t))^{\alpha} + q(t)f(u(t)) = \Re(t,u(t),u'(t)), \tag{1.3}$$

$\forall\ t > 0$, with initial conditions

$$u(t_0) = u'(t_0) = 0. \tag{1.4}$$

where $r(t),\ p(t), q(t) \in C(\mathbf{R}_+)$, $\psi,\ f \in C(\mathbf{R})$, $\Bbbk, \Re \in C(\mathbf{I} \times \mathbf{R^2}, \mathbf{R})$, $\alpha \in \mathbf{N}$ (set of posive whole numbers) and $r(t) > 0$.

In 1998, Alsina and Ger [11] investigated the HUS of an ordinary differential equation

$$u'(t) = u(t).$$

Miura *et al.* [12], Takahasi *et.al* [13], and Miura *et al.* [14] expanded on the findings of Alsina and Ger[11], demonstrating that the HUS is maintained for the ordinary differential equation

$$u'(t) = \lambda u(t),$$

while Jung [15] proved a similar result for equation

$$\psi(t)u'(t) = u(t).$$

In addition, for first-order linear ordinary differential equations, Miura *et al.*[16], Takahasi *et al.* [17], and Jung [18] have generalized the Hyers-Ulam result. They addressed the first-order nonhomogeneous linear differential equation.

$$u'(t) + p(t)u(t) + q(t) = 0.$$

Jung [19] recently demonstrated that differential equation of the form

$$tu'(t) + \alpha u(t) + \beta t^r x_0 = 0$$

have generalized HUS and also used this result to consider the HUS of Euler (Cauchy) differential equation

$$t^2u''(t) + \alpha t u'(\xi) + \beta u(t) = 0.$$

Algfiary and Jung [20], Qarawani [21, 22], Fakunle and Arawomo [23, 24, 25, 26, 27, 28, 29] and Rus [30,31] are among of the few writers who have written on HUS of nonlinear differential equations.The work of Fakunle and Arawomo [24] served as the inspiration for this investigation.

2. Preliminary

Definitions, lemmas, and theorems listed below are useful in this work.

Definition 1 If $\epsilon > 0$ and a solution $u(t) \in C^2(\mathbf{R}_+)$, exists such that

$$|(r(t)\psi(u(t))(u'(t))^{\alpha})' + p(t)(u'(t))^{\alpha} + q(t)f(u(t)) - \Re(t,u(t),u'(t))| \leq \epsilon. \tag{2.1}$$

Furthermore, any solution $u_0(t) \in C^2(\mathbf{R}_+)$ of (1.1) exists such that

$$|u(t) - u_0(t)| \leq K\epsilon,$$

for K is positive Hyers-Ulam-constant (HUC).

Definition 2 Equation (1.2) has HUS, if there $\exists$ a constant $K \geq 0$ and for every $\epsilon \geq 0$, and $u(t) \in C^2(\mathbf{R}_+)$, that satisfy

$$\begin{aligned} &|(r(t)\psi(u(t))(u'(t))' + p(t)\Bbbk(t,u(t),u'(t))(u'(t)) + q(t)f(u(t)) \\ &-\mathfrak{R}(t,u(t),u'(t))| \leq \epsilon, \end{aligned} \tag{2.2}$$

then, if $u_0(t) \in C^2(\mathbf{R}_+)$ is any solution satisfying (1.2) such that

$$|u(t) - u_0(t)| \leq K\epsilon,$$

where K is HUC of equation (1.2).

Definition 3 We say that (1.3) is HUS with initial conditions, if there exists a positive constant K with the property: For every $\epsilon > 0$, $u(t) \in C^2(\mathbf{R}_+)$, if

$$|(r(t)(u'(t))' + p(t)(u'(t))^\alpha + q(t)f(u(t)) - \mathfrak{R}(t,u(t),u'(t))| \leq \epsilon, \tag{2.3}$$

then there exists solution $u_0(t) \in C^2(\mathbf{R}_+)$ satisfying (1.3) such that

$$|u(t) - u_0(t)| \leq K\epsilon,$$

where K is Hyers-Ulam constant.

Definition 4 A function $\Lambda: [0,\infty) \to [0,\infty)$ is said to belong to a class S if

i. $\Lambda(\varepsilon)$ is nondecreasing and continuous for $\varepsilon \geq 0$

ii. $(\frac{1}{\upsilon})\Lambda(\varepsilon) \leq \Lambda(\frac{\varepsilon}{\upsilon})$ for all $\varepsilon \quad and \quad \upsilon \geq 1.$

iii. there exist a function ϕ, continuous on $[0,\infty)$ with $\Lambda(\alpha\varepsilon) \leq \phi(\alpha)\Lambda(\upsilon)$ for $\alpha \geq 0$.

Theorem 1 [32] Let

i. $\varepsilon(t), \iota(t): (0,\infty) \to (0,\infty)$ and continuous on $(0,\infty)$,

ii. $\psi \in S$,

iii. $n > 0$ be monotonic, nondecreasing and continuous on $(0,\infty)$,

if

$$\varepsilon(t) \leq n(t) + \int_0^t \eta(s)\phi(\varepsilon(s))ds, \;\; 0 < t < \infty, \tag{2.4}$$

then

$$\begin{aligned} &\varepsilon(t) \leq n(t)\Omega^{-1}\left(\Omega(1) + \int_0^t \eta(s)ds\right), \\ &\;0 < t \leq b, \end{aligned} \tag{2.5}$$

where

$$\Omega(\varepsilon) = \int_{\varepsilon_0}^{\varepsilon} \frac{dt}{\phi(t)}, \;\; 0 < \varepsilon_0 < \varepsilon. \tag{2.6}$$

Remark: Theorem 1, has its foundation from [33, 34] and application in [35].

Theorem 2 [36] For every $\eta(t)\quad and \quad \rho(t)$ that is continuous in $[t_0, t] \subseteq \mathbf{I}$, there exists a point $\zeta \in [t_0, t]$ such that $\int_{t_0}^{t} \rho(s)\eta(s)ds = \rho(\zeta)\int_{t_0}^{t} \eta(s)ds$.

Theorem 3 [24, 25] Assume that $\phi(\varepsilon)$ is submultiplicative for $\varepsilon > 0$ and that $\phi(\varepsilon), \beta(\varepsilon)$ be long to class S and $\varepsilon(t), \iota(t), \kappa(t) \in C(\mathbf{I}, \mathbf{R}_+)$ are nonnegative, monotonic, nondecreasing, continuous functions. Let

$$\varepsilon(t) \leq \Im + T\int_{t_0}^{t} \iota(s)\beta(\varepsilon(s))ds + L\int_{t_0}^{t} \kappa(s)\phi(\varepsilon(s))ds \qquad (2.7)$$

for $\Im,\ T\ and\ L > 0$, then

$$\begin{aligned}&\varepsilon(t) \leq \Omega^{-1}\Big(\Omega(\Im) + L\int_{t_0}^{t} \kappa(s)\phi(F^{-1}(F(1)\\ &+T\int_{t_0}^{s} \iota(\alpha)d\alpha\Big)\Big)ds\Big) \qquad (2.8)\\ &F^{-1}\Big(F(1) + T\int_{t_0}^{t} \iota(s)ds\Big)\end{aligned}$$

where $\beta(\varepsilon) \neq \phi(\varepsilon)$, Ω is defined in equation (2.6) and

$$F(\varepsilon) = \int_{\varepsilon_0}^{\varepsilon} \frac{ds}{\beta(s)}, \quad 0 < \varepsilon_0 \leq \varepsilon, \qquad (2.9)$$

F^{-1}, Ω^{-1} are the inverses of F, Ω respectively and ξ is in the subinterval $(0, b) \in \mathbf{I}$

Corollary 1 [24, 25] Suppose $\chi(t)$ is nonnegative, monotonic, nondecreasing continuous function on $\mathbf{R}_+$. Let

$$\varepsilon(t) \leq \chi(t) + T\int_{t_0}^{t} \iota(s)\beta(u(s))ds + L\int_{t_0}^{t} \kappa(s)\phi(u(s))ds, \quad (2.10)$$

for $T\quad and \quad L$ are positive constants, then

$$\begin{aligned}&\varepsilon(t) \leq \chi(t)\Omega^{-1}\Big(\Omega(1) + L\int_{t_0}^{t} \kappa(s)\phi\Big(F^{-1}\Big(F(1) + T\int_{t_0}^{t} \iota(\alpha)d\alpha\Big)\Big)ds\Big)\\ &F^{-1}\Big(F(1) + T\int_{t_0}^{t} \iota(s)ds\Big), \quad t \in \mathbf{I},\end{aligned}$$

where $\Omega(\varepsilon)$ and $F(\varepsilon)$ are defined as in (2.6) and (2.9) respectively.

Theorem 4 [24, 25] Let $\varepsilon(t), \iota(t), \kappa(t), \rho(t): \mathbf{I} \to \mathbf{R}_+$ be real valued nonnegative continuous functions and $\phi(\varepsilon), \eta(\varepsilon), \gamma(\varepsilon)$ be positive, monotonic, nondecreasing continuous functions on $\mathbf{R}_+$ and let these functions belong to class of S. If $\gamma(\varepsilon)$ is submultiplicative for $\varepsilon > 0$ and $A,\ T,\ L$ are positive constants. If

$$\begin{aligned}&\varepsilon(t) \leq \chi(t) + A\int_{t_0}^{t} \iota(s)\eta(\varepsilon(s))ds + T\int_{t_0}^{t} \kappa(s)\phi(\varepsilon(s))ds\\ &+L\int_{t_0}^{t} \rho(s)\gamma(\varepsilon(s))ds, \quad t \in \mathbf{I},\end{aligned} \qquad (2.11)$$

then

$$\begin{aligned}&\varepsilon(t) \leq \chi(t)G^{-1}\Big[G(1) + L\int_{t_0}^{t} \rho(s)\gamma[\Omega^{-1}(V(s)F^{-1}(B(s)))]ds\Big]\\ &\Omega^{-1}(V(t))F_{-1}(B(t)),\end{aligned}$$

where F, is defined in (2.9) and

$$V(t) = \Omega(1) + T\int_{t_0}^{t} \kappa(s)\phi(F^{-1}(B(s)))ds \quad (2.12)$$

and

$$B(t) = F(1) + A\int_{t_0}^{t} \iota(s)ds. \qquad (2.13)$$

Function G is defined as

$$G(\varpi) = \int_{\varpi_0}^{\varpi} \frac{ds}{\gamma(s)} \quad 0 < \varpi_0 \leq \varpi, \qquad (2.14)$$

and $F^{-1}, \ \Omega^{-1} \ and \ G^{-1}$ are the inverses of the functions $F, \ \Omega$, and G respectively.

Lemma 1 [37] Let $\iota(t)$ be an integrable function then the n-successive integration of ι over the interval $[t_0, t]$ is given by

$$\int_{t_0}^{t} \dots \int_{t_0}^{t} \iota(s)ds^n = \frac{1}{(n-1)!}\int_{t_0}^{t} (t-s)^{n-1}\iota(s)ds. \qquad (2.15)$$

3. Results and Discussion

Theorem 5 Let $r(t), p(t), \ \phi(t) \ and \ q(t)$ be nondecreasing, continuous functions on $\mathbf{R}_{+}$, such that

i. $\Re(t, u(t), u'(t)) = \phi(t)\varpi(u(t))(u'(t))^n$ where $n \in \mathbf{Z}_{+}$.

ii. set $F(u(t)) = \int_{u(t_0)}^{u(t)} f(s)ds$

iii. let f be differentiable on interval **I**, if $f'(t) \geq 0 \ forallt \in \mathbf{I}$, then f is non

decreasing on **I** and $f(t) \geq \sigma$ where σ a positive constant.

iv. let $|F(u(t))| \geq |u(t)|$

v. $|u'(t)| \leq \lambda$ where $\lambda > 0$,

vi. $\lim_{t_0 \to \infty} \int_{t_0}^{t} r(s)ds \leq d_1 < \infty, \ where \ d_1 > 0,$

vii. $\lim_{t_0 \to \infty} \int_{t_0}^{t} p(s)ds \leq d_2 < \infty, \ where \ d_2 > 0,$

viii. $\lim_{t_0 \to \infty} \int_{t_0}^{t} \phi(s)ds \leq d_3 < \infty, \ where \ d_3 > 0,$

ix. $\int_{t_0}^{\infty} |u'(s)|ds \leq L < \infty, \ where \ L > 0,$

are satisfied. In addition, let $\varpi(u(t)) \in S$ be continuous, nondecreasing and monotonic, then, (1.1) has the HUS with HUC.

Proof. We begin the proof from inequality (2.1) by multiplying with $u'(t)$, integrating twice, applying Lemma1, conditions (i)-(v) of Theorem 5 and Theorem 2 implies there exist $\eta, \rho, \xi_1 \in [t_0, t]$ such that

$$|u(t)| \leq \frac{\left(\int_{t_0}^{t} |u\prime(s)|ds + (|u\prime(\rho)|)^{\alpha+1}\int_{t_0}^{t} p(s)ds\right)}{\sigma}\epsilon$$
$$+ \frac{\lambda(|u\prime(\eta)|)^{\alpha}}{\sigma}\int_{t_0}^{t} (r(s)\psi(|u(s)|)ds + \frac{(|u\prime(\xi_1)|)^{n+1}}{\sigma}\int_{t_0}^{t} \phi(s)\varpi(|u(s)|)ds, \quad t > 0.$$

Corollary 1 is used to obtained

$$
\begin{aligned}
&|u(t)| \leq \frac{\left(\int_{t_0}^{t} |u\prime(s)|ds + (|u\prime(\rho)|)^{\alpha+1} \int_{t_0}^{t} p(s)ds\right)}{\sigma}\epsilon \\
&\Omega^{-1}\left(\Omega(1) + \frac{(|u\prime(\xi_1)|)^{n+1}}{\sigma}\int_{t_0}^{t} \phi(s)\varpi\left(F^{-1}\left(F(1) + \frac{|u\prime(\varepsilon)|(|u\prime(\eta)|)^{\alpha}}{\sigma}\int_{t_0}^{t} r(s)ds\right)\right)ds\right) \qquad (3.1) \\
&F^{-1}\left(F(1) + \frac{|u\prime(\varepsilon)|(|u\prime(\eta)|)^{\alpha}}{\sigma}\int_{t_0}^{t} r(s)ds\right)
\end{aligned}
$$

conditions (vi)- (ix) of Theorem 5 are employed to get

$$
\begin{aligned}
&|u(t)| \leq \frac{(L + d_2(|u\prime(\rho)|)^{\alpha+1})}{\sigma}\epsilon \\
&\Omega^{-1}\left(\Omega(1) + \frac{d_3(|u\prime(\xi_1)|)^{n+1}}{\sigma}\varpi\left(F^{-1}\left(F(1) + \frac{d_1\lambda(|u\prime(\eta)|)^{\alpha}}{\sigma}\right)\right)\right) \\
&F^{-1}\left(F(1) + \frac{d_1\lambda(|u\prime(\eta)|)^{\alpha}}{\sigma}\right)
\end{aligned}
$$

Hence,

$$|u(t) - u(t_0)| \leq |u(t)| \leq K_1\epsilon,$$

where

$$
\begin{aligned}
&K_1 = \frac{(L + d_2(|u\prime(\rho)|)^{\alpha+1})}{\sigma} \\
&\Omega^{-1}\left(\Omega(1) + \frac{d_3(|u\prime(\xi_1)|)^{n+1}}{\sigma}\varpi\left(F^{-1}\left(F(1) + \frac{d_1\lambda(|u\prime(\eta)|)^{\alpha}}{\sigma}\right)\right)\right) \\
&F^{-1}\left(F(1) + \frac{d_1\lambda(|u\prime(\eta)|)^{\alpha}}{\sigma}\right)
\end{aligned}
$$

Theorem 6 Let the conditions (i)-(viii) of Theorem 5 hold. In addition, the following conditions:

i $\Bbbk(t, u(t), u'(t)) = v(t)g(u(t))h(u'(t))$, where $v(t)$ a monotonic continuos function on $\mathbf{R}_+$

ii. $\lim\limits_{t_0\to\infty} \int_{t_0}^{t} p(s)v(s)ds \leq d_4 < \infty,\ where\ \ d_4 > 0,$

are satisfied, where $g(u(t)) \in S$ is continuous, nondecreasing and monotonic function then equation (1.2) has the HUS with HUC.

Proof. Multiplying inequality (2.2) by $u'(t)$, applying Lemma 1, conditions (i)-(ii) of Theorem 5 and (ii') of Theorem 6 we obtain

$$
\begin{aligned}
&u'(t)\int_{t_0}^{t} (r(s)\psi(u(s))(u'(t))^{\alpha})ds + (u'(t))^2\int_{t_0}^{t} p(s)v(s)g(u(s))h(u'(s))ds \\
&+ \int_{t_0}^{t} q(s)\frac{d}{ds}F(u(s))ds - \int_{t_0}^{t} \phi(s)\varpi(u(s))(u'(s))^{n+1}ds \leq \epsilon\int_{t_0}^{t} u'(s)ds.
\end{aligned}
$$

Using the following conditions (iii)-(v) of Theorem 5 and Theorem 2 let there be $\xi_1, \beta.\gamma \in [t_0, t]$ such that

$$
\begin{aligned}
&|u(t)| \leq \frac{\epsilon}{\sigma}\int_{t_0}^{t} |u'(s)|ds + \frac{(|u\prime(\xi_1)|)^{\alpha}\lambda}{\sigma}\int_{t_0}^{t} (r(s)\psi(|u(s)|)ds \\
&+ \frac{h(|u\prime(\gamma)|)\lambda^2}{\sigma}\int_{t_0}^{t} p(s)v(s)g(|u(s)|)ds \\
&+ \frac{|u\prime(\beta)|)^{n+1}}{\sigma}\int_{t_0}^{t} \phi(s)\varpi(|u(s)|)ds.
\end{aligned}
$$

Using Theorem 4 to obtain

$$|u(t)| \leq \epsilon \int_{t_0}^{t} |u'(s)|ds$$
$$G^{-1}\left[G(1) + \frac{(|u\prime(\beta)|)^{n+1}}{\sigma}\int_{t_0}^{t} \phi(s)\varpi[\Omega^{-1}(V(s)F^{-1}(B(s)))]ds\right]$$
$$\Omega^{-1}(V(t))F^{-1}(B(t)),$$

where

$$V(t) = \Omega(1) + \frac{h(|u\prime(\gamma)|)\lambda^2}{\sigma}\int_{t_0}^{t} p(s)v(s)g(F^{-1}(B(s)))ds$$

and

$$B(t) = F(1) + \frac{(|u\prime(\xi_1)|)^{\alpha}\lambda}{\sigma}\int_{t_0}^{t} r(s)ds$$

Applying conditions (vi)-(viii) of Theorem 5 and (ii’) of Theorem 6, we arrive at

$$|u(t)| \leq \epsilon L$$
$$G^{-1}\left[G(1) + d_3\frac{(|u\prime(\beta)|)^{n+1}}{\sigma}\varpi[\Omega^{-1}(V^*F^{-1}(B^*))]\right]$$
$$\Omega^{-1}(V^*)F^{-1}(B^*),$$

where

$$V^* = \Omega(1) + d_4\frac{h(|u\prime(\gamma)|)\lambda^2}{\sigma}g(F^{-1}(B^*))$$

and

$$B^* = F(1) + d_1\frac{(|u\prime(\xi_1)|)^{\alpha}\lambda}{\sigma}$$

Hence,

$$|u(t) - u(t_0)| \leq |u(t)| \leq K_2\epsilon,$$

where

$$K_2 = LG^{-1}\left[G(1) + d_3\frac{(|u\prime(\beta)|)^{n+1}}{\sigma}\varpi[\Omega^{-1}(V^*F^{-1}(B^*))]\right]$$
$$\Omega^{-1}(V^*)F^{-1}(B^*).$$

Theorem 7 Let the conditions (i)-(iv),(vi)-(ix) of the Theorem 5 remain valid, then equation (1.3) has HUS with HUC.

***Proof*.** We use (2.3) together with application of conditions (i)-(iv) of Theorem 5 and Theorem 2 let be $\rho,\ \xi_1, \tau \in [t_0, t]$ such that such that

$$|u(t)| \leq \frac{\left((|u\prime(\rho)|)^2\int_{t_0}^{t} r(s)ds+(|u\prime(\xi_1)|)^{\alpha+1}\int_{t_0}^{t} p(s)ds+\int_{t_0}^{t} |u\prime(s)|ds\right)}{\sigma}\epsilon$$
$$+\frac{|u\prime(\tau)|^{n+1}}{\sigma}\int_{t_0}^{t} \phi(s)\varpi(|u(s)|)ds, \quad t > 0.$$

Use Theorem 1 to arrive at

$$|u(t)| \leq \epsilon\frac{\left((|u\prime(\rho)|)^2\int_{t_0}^{t} r(s)ds+(|u\prime(\xi_1)|)^{\alpha+1}\int_{t_0}^{t} p(s)ds+\int_{t_0}^{t} |u\prime(s)|ds\right)}{\sigma}$$
$$\Omega^{-1}\left(\Omega(1) + \frac{|u\prime(\tau)|^{n+1}}{\sigma}\int_{t_0}^{t} \phi(s)ds\right).$$

Using conditions (vi)-(ix) of Theorem 5 we obtain

$$|u(t)-u(t_0)| \leq |u(t)| \leq \epsilon \frac{\left(d_1(|u\prime(\rho)|)^2+d_2(|u\prime(\xi_1)|)^{\alpha+1}+L\right)}{\sigma}$$
$$\Omega^{-1}\left(\Omega(1)+\frac{|u\prime(\tau)|^{n+1}}{\sigma} d_3\right).$$

where

$$K_3=\frac{\left(d_1(|u\prime(\rho)|)^2+d_2(|u\prime(\xi_1)|)^{\alpha+1}+L\right)}{\sigma}$$
$$\Omega^{-1}\left(\Omega(1)+\frac{|u\prime(\tau)|^{n+1}}{\sigma} d_3\right).$$

Theorem 8 Supposed conditions(ii)-(vii),(ix) of Theorem 5 remain valid. If $\Re(t,u(t),u'(t))=0$ in (1.1) that is,

$$(r(t)\psi(u(t))(u'(t))^{\alpha})'+p(t)(u'(t))^{\alpha}+q(t)f(u(t))=0, \quad (3.2)$$

then,(3.2) has HUS with HUC.

***Proof*.** Consider (2.1), if $\Re(t,u(t),u'(t))=0$, by further evaluation, using Lemma 1, conditions (ii)-(iii) of Theorem 5 and Theorem 2 there exist $\xi_1, \rho \in [t_0,t]$ such that

$$\sigma F(u(t)) \leq \epsilon \int_{t_0}^{t} u'(s)ds-\xi(u'(\rho))^{\alpha+1}\int_{t_0}^{t} p(s)ds$$
$$-(u'(t))(u'(\xi_1))^{\alpha}\int_{t_0}^{t} (r(s)\psi(u(s))ds, \quad t>0.$$

By conditions (iv)-(v) of Theorem 5 we get

$$|u(t)| \leq \frac{\left(\int_{t_0}^{t}|u\prime(s)|ds+(|u\prime(\rho)|)^{\alpha+1}\int_{t_0}^{t}p(s)ds\right)}{\sigma}\epsilon+\frac{\lambda(|u\prime(\xi_1)|)^{\alpha}}{\sigma}\int_{t_0}^{t} (r(s)\psi(|u(s)|)ds.$$

Let us make use of Theorem 1 to obtain

$$|u(t)| \leq \frac{\left(\int_{t_0}^{t}|u\prime(s)|ds+(|u\prime(\rho)|)^{\alpha+1}\int_{t_0}^{t}p(s)ds\right)}{\sigma}\epsilon\Omega^{-1}\left(\Omega(1)+\frac{\lambda(|u\prime(\xi_1)|)^{\alpha}}{\sigma}\int_{t_0}^{t} (r(s)ds\right),$$

and using conditions (vi),(vii),(ix) of Theorem 5 to arrive at

$$|u(t)| \leq \frac{(L+d_2(|u\prime(\rho)|)^{\alpha+1})}{\sigma}\epsilon\Omega^{-1}\left(\Omega(1)+d_1\frac{\lambda(|u\prime(\xi_1)|)^{\alpha}}{\sigma}\right).$$

where

$$K_4=\frac{(L+d_2(|u\prime(\rho)|)^{\alpha+1})}{\sigma}\Omega^{-1}\left(\Omega(1)+d_1\frac{\lambda(|u\prime(\xi_1)|)^{\alpha}}{\sigma}\right).$$

Theorem 9 Let the conditions (ii)-(iv),(vi),(ix) of Theorem 5 and (i'),(ii') of the Theorem 6 hold. Equation

$$(r(t)\psi(u(t))(u'(t))^{\alpha})'+p(t)\Bbbk(t,u(t),u'(t))(u'(t))+q(t)f(u(t))=0 \qquad (3.3)$$

has the HUS with HUC.

***Proof*.** Setting $\Re(t,u(t),u'(t))=0$ in inequality (2.2), integrating twice, applying Lemma 1,condition (ii) of Theorem 5 and Theorem 2 implies there exists $\xi_1 \in [t_0,t]$ such that

$$(u'(t))(u'(\xi_1))^{\alpha}\int_{t_0}^{t} (r(s)\psi(u(s))ds+(u'(\xi_1))^2\int_{t_0}^{t} p(s)\Bbbk(s,u(s),u'(s))ds$$
$$+q(t)F(u(t)) \leq \epsilon\int_{t_0}^{t} u'(s)ds, >0.$$

Let us make use of conditions (iii)-(v) of Theorem 5, (i') of Theorem 6 with the application of Theorem 2 there exists $\gamma \in [\xi_0,\xi]$ such that

$$|u(t)| \leq \frac{\epsilon}{\sigma}\int_{t_0}^{t} |u'(s)|ds + \frac{\lambda(|u'(\xi_1)|)^{\alpha}}{\sigma}\int_{t_0}^{t} (r(s)\psi(|u(s)|)ds$$
$$+\frac{\lambda^2 h(|u'(\gamma)|)}{\sigma}\int_{t_0}^{t} p(s)v(s)g(|u(s)|)ds.$$

Applying Corollary1, we obtain

$$|u(t)| \leq \frac{\epsilon}{\sigma}\int_{t_0}^{t} |u'(s)|ds\Omega^{-1}\left(\Omega(1) + \frac{\lambda^2 h(|u'(\gamma)|)}{\sigma}\int_{t_0}^{t} p(s)v(s)\right.$$
$$\left.g\left(F^{-1}\left(F(1) + \frac{\lambda(|u'(\xi_1)|)^{\alpha}}{\sigma}\int_{t_0}^{t} r(\alpha)d\alpha\right)\right)ds\right)$$
$$F^{-1}\left(F(1) + \frac{\lambda(|u'(t)|)^{\alpha}}{\sigma}\int_{t_0}^{t} r(s)ds\right), \quad t \in \mathbf{I},$$

Applying conditions (vi),(ix) of Theorem 5 and (ii') of Theorem 6, we arrive at

$$|u(t)| \leq \epsilon\frac{L}{\sigma}\Omega^{-1}\left(\Omega(1) + d_4\frac{\lambda^2 h(|u'(\gamma)|)}{\sigma}\right.$$
$$\left.g\left(F^{-1}\left(F(1) + d_1\frac{\lambda(|u'(\xi_1)|)^{\alpha}}{\sigma}\right)\right)\right)F^{-1}\left(F(1) + \frac{\lambda(|u'(\xi_1)|)^{\alpha}}{\sigma}d_1\right).$$

Hence,

$$|u(t) - u(t_0)| \leq |u(t)| \leq K_5\epsilon,$$

where

$$K_5 = \frac{L}{\sigma}\Omega^{-1}\left(\Omega(1) + d_4\frac{\lambda^2 h(|u'(\gamma)|)}{\sigma}\right.$$
$$\left.g\left(F^{-1}\left(F(1) + d_1\frac{\lambda(|u'(\xi_1)|)^{\alpha}}{\sigma}\right)\right)\right)F^{-1}\left(F(1) + \frac{\lambda(|u'(\xi_1)|)^{\alpha}}{\sigma}d_1\right).$$

Theorem 10 Conditions (ii)-(vii) of the Theorem 5 remain satisfied, then equation

$$(r(t)u'(t))' + p(t)(u'(t))^{\alpha}) + q(t)f(u(t)) = 0, \qquad (3.4)$$

has HUS with HUC.

Proof. From (2.9), if $\Re(t, u(t), u'(t)) = 0$, integrating twice, using Lemma 1 and conditions (ii), (iii) of Theorem 5 it is clear that

$$u'(t)\int_{t_0}^{t} (r(s)(u'(s))ds + \int_{t_0}^{t} p(s)(u'(s))^{\alpha+1}ds + \sigma F(u(t)) \leq \epsilon\int_{t_0}^{t} u'(s)ds, \quad \forall t > 0.$$

We use Theorem 2 there exists $\mu,\ \rho \in [t_0, t]$ such that

$$\sigma F(u(\xi)) \leq \epsilon\int_{\xi_0}^{\xi} u'(s)ds - (u'(\xi)(u'(\mu)))\int_{t_0}^{t} r(s)ds - (u'(\rho))^{\alpha+1}\int_{t_0}^{t} p(s)ds, \quad \forall t > 0.$$

Using the conditions(iv)-(v) of Theorem5 we have

$$|u(t)| \leq \epsilon\left(\int_{t_0}^{t} |u'(s)|ds + (|u'(\xi_1)|)^{\alpha+1}\int_{t_0}^{t} p(s)ds + \lambda|u'(\mu)|\int_{t_0}^{t} r(s)ds\right).$$

Furthermore, we use conditions (vi)-(vii) of Theorem 5 to arrive at

$$|u(t) - u(t_0)| \leq |u(t)| \leq \epsilon(L + d_2(|u'(t)|)^{\alpha+1} + d_1\lambda|u'(\mu)|).$$

Hence,

$$|u(t) - u(t_0)| \leq K_6\epsilon,$$

where

$$K_6 = (L + d_2(|u'(t)|)^{\alpha+1} + d_1\lambda|u'(\mu)|).$$

Example 1 Investigate the HUS of the following nonlinear differential equation

$$\left(\frac{1}{t^6}u^4(t)(u'(t))^{\alpha}\right)' + \frac{1}{t^4}(u'(t))^{\alpha} + \frac{1}{t^3}u^2(t) = \frac{1}{t^5}u^4(t)u'^3(\xi),$$

where α a positive integer, that is $\alpha \geq 1$, $r(t) = \frac{1}{t^6}$, $p(t) = \frac{1}{t^4}$, $q(t) = \frac{1}{t^3}$ and $\Re(t, u(t), u'(t)) = \frac{1}{t^5}u^4(t)u'^3(t)$, using inequality(3.1) in the proof of Theorem 5 that is

$$\begin{aligned}|u(t)| \leq{} & \frac{\left(\int_{t_0}^{t}|u'(s)|ds + (|u'(\rho)|)^{\alpha+1}\int_{t_0}^{t}\frac{1}{s^4}ds\right)}{\sigma}\epsilon \\ & \Omega^{-1}\left(\Omega(1) + \frac{(|u'(\xi_1)|)^{n+1}}{\sigma}\int_{t_0}^{t}\frac{1}{s^5}\varpi\left(F^{-1}\left(F(1) + \frac{|u'(\varepsilon)|(|u'(\eta)|)^{\alpha}}{\sigma}\int_{t_0}^{t}\frac{1}{s^6}ds\right)\right)ds\right) \\ & F^{-1}\left(F(1) + \frac{|u'(\varepsilon)|(|u'(\eta)|)^{\alpha}}{\sigma}\int_{t_0}^{t}\frac{1}{s^6}ds\right)\end{aligned}$$

The result is easily obtain using the following:

1. $\lim\limits_{t_0\to\infty}\int_{t_0}^{t}\frac{1}{s^6}ds \leq d_1 < \infty,\ where\ \ d_1 > 0,$
2. $\lim\limits_{t_0\to\infty}\int_{t_0}^{t}\frac{1}{s^4}ds \leq d_2 < \infty,\ where\ \ d_2 > 0,$
3. $\lim\limits_{t_0\to\infty}\int_{t_0}^{t}\frac{1}{s^5}ds \leq d_3 < \infty,\ where\ \ d_3 > 0,$
4. $\int_{t_0}^{\infty}|u'(s)|ds \leq L < \infty,\ where\ \ L > 0,$

that

$$\begin{aligned}|u(t)| \leq{} & \frac{\left(L + d_2(|u'(\rho)|)^{\alpha+1}\right)}{\sigma}\epsilon \\ & \Omega^{-1}\left(\Omega(1) + \frac{d_3(|u'(\xi_1)|)^{n+1}}{\sigma}\varpi\left(F^{-1}\left(F(1) + \frac{d_1\lambda(|u'(\eta)|)^{\alpha}}{\sigma}\right)\right)\right) \\ & F^{-1}\left(F(1) + \frac{d_1\lambda(|u'(\eta)|)^{\alpha}}{\sigma}\right)\end{aligned}$$

where

$$\begin{aligned}K ={} & \frac{\left(L + d_2(|u'(\rho)|)^{\alpha+1}\right)}{\sigma} \\ & \Omega^{-1}\left(\Omega(1) + \frac{d_3(|u'(\xi_1)|)^{n+1}}{\sigma}\varpi\left(F^{-1}\left(F(1) + \frac{d_1\lambda(|u'(\eta)|)^{\alpha}}{\sigma}\right)\right)\right) \\ & F^{-1}\left(F(1) + \frac{d_1\lambda(|u'(\eta)|)^{\alpha}}{\sigma}\right)\end{aligned}$$

Example 2 Investigate the HUS of the following nonlinear differential equation

$$\frac{1}{t^2}(u'(t))' + \frac{1}{t^4}(u'(t))^{\alpha} + \frac{1}{3}u^2(t) = \frac{1}{t^5}u^4(t)u'^3(t),$$

where α is a positive integer, that is $\alpha \geq 1$, $r(t) = \frac{1}{t}$, $p(t) = \frac{1}{t^4}$, $q(t) = \frac{1}{t^3}$ and $\Re(t, u(t), u'(t)) = \frac{1}{t^5}u^4(t)u'^3(t)$, using Theorem 7, we follow the same principle used in the first example to arrive at the result.

4. Conclusion

Some problems such as hereditary problem, population dynamics, the surge in birth-rate, smoke filtration in cigarette and so on appear directly in terms of differential equations which can be transform to integral equations. One of the ways of considering the stability of such equations is via HUS of nonlinear differential equation involved.

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