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Article

Numerical Solution of Optimization Problems Via Multiplier Method

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Abstract

This paper discusses the use of conjugate gradient algorithm and Modified Newton's method employed to determine the Solution of equality constrained optimization problems. The conjugate gradient algorithm was used as a scaling factor with the purpose of making the initial guess to be closer to the optimal solution, after which the Newton's method was introduced to guide against jumping the optimal points and ill-conditioning of the problems along the search path. A Lagrange multiplier Vector updating scheme at each one-dimensional search is considered. It was proved using some tested problems that the rate of convergence of the method is linear and lesser number of iterations will be generated if one starts with sufficiently small penalty factor.

Keywords: Conjugate Gradient Algorithm, Modified Newton's Method, Multiplier Method, Numerical Solution, Optimization Problems.

1. Introduction

We studied a class of methods for solving constrained optimization problems which are based on quadratic Augmented Lagrangian for which the penalty parameters are functions of multipliers [1]. This gives rise to Lagrangian which are Nonlinear in the multipliers. We equally studied some method for unconstrained multivariate problems such as conjugate gradient algorithm and the Modified Newton's methods. The conjugate gradient algorithm considered the steepest direction and an updated search direction while the Modified Newton's method considered the inverse Hessian matrix of the function under consideration [2].

For unconstrained problems, the two methods are very suitable, though the conjugate gradient method is preferred because it converges faster than the Newton's method. If a problem is subject to one or more constraints, experience reveals that the iterates of conjugate gradient algorithm may move away from the feasible region in its quest to locate an optimal path in short time, thereby repeating the same path before locating a local optimal point. This action of the algorithm waste computer time, which is not a characteristic of a good algorithm [3]. On the other hand, the Modified Newton's method moves slowly in search for global minimum thereby generate more iteration [4]. In order to balance the two sides, the two methods are combined and applied as an unconstrained algorithm for the multiplier methods.

Since the scaled multiplier method is to be used in conjunction with the conjugate gradient, and the modified Newton's method is to be used along with the conjugate gradient and modified Newton's method, these methods are briefly outlined here. The conjugate method uses the steepest direction together with the Hessian matrix of the functions to be optimized [5] and [6] and the Modified Newton's method uses both the steepest direction as well as the inverse matrix of the function to be optimized [7].

The rest of the paper consists of the following sections: Definition of Terms, Multiplier Method, Modified Multiplier Method, Scaled Multiplier Method, Analysis of Data, Comment and Conclusions.

1.1. Some Useful Terms

Optimization: is the act of obtaining the best option under some given set of circumstances.
Feasible Solution: This is the value of the variables that satisfies all set of given constraints.
Feasible Region: This is the set of all feasible solutions.
Optimal solution: This is the most favorable or best feasible solution.

Definition 1.1: Local Minimum: A function $f(x)$ has a local minimum at the value x^* if there exists neighborhood of x^* such that $\forall$ values of x in this neighborhood $f(x)$ is at least as large as $f(x^*)$. It may simply be defined as the lowest function value in a finite neighborhood but not on neighborhood boundary.

Definition 1.2: Strict Local Minimum: Let $f(x)$ be a function, if $f(x) < f(x^*)$ $\forall x$ and $x \neq x^*$ $\forall x$ Subject to $|x - x^*| < \varepsilon$, then x^* is said to be strict (relative) local minimum point of f.

Definition 1.3: Global Minimum: Function $f(x)$ has a global minimum at point x^* if $f(x) \geq f(x^*)$. i.e. a function $f(x^*)$ $\forall x$ is a global minimum if $f(x^*)$ has lowest minimum value.

Definition 1.4: Optimal Value: This is the value of the objective function that corresponds to an optimal solution.

2. The Main Results

2.1 Multiplier and Lagrange Multiplier Methods

We shall start with a brief overview of the multiplier method for the constrained problem presented in E^n:

$$\begin{aligned}&Minimize\ \phi = \phi(x),\ x \in E^n \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)\\&subject\ to\ \ h(x) = 0\\&Where\, is\ h\ an\ m - vector,\ \ m < n\end{aligned}$$

This method combines the penalty function approach with the Lagrange multiplier technique [8].

The penalty function method focuses on minimizing:

$$\varphi^i(x) = \phi(x) + \mu h^T(x)h(x).$$

As the scalar values μ^i increase, the limit of the minima for these problems as $\mu^i \to \infty$, if it exists, yields the solution (1).

In contrast, the Lagrange multiplier method examines a function:

$$\psi(x) = \phi(x) + \lambda^T h(x).$$

Where the m-vector λ is derived from simultaneously solving the system of equations

$$\phi_x(x) + h_x^T(x)\lambda = 0$$

$$h(x) = 0 \quad \text{..(2)}$$

It involves $n-$ vector x and the $m-$ vector λ.

The multiplier method evaluates the function $\psi^i(x)$ at each step:

$$\psi^i(x) = \varphi(x) + \frac{1}{2}\mu h^T(x)h(x) + \lambda^{iT} h(x)$$

and seeks to minimize it.

After each iteration of minimization, the Lagrange multiplier vector, λ^i (as opposed to the penalty coefficient μ), is updated [11]. At the minimum point of $\psi^i(x)$, we arrive at:

$$\phi_x(x^i) + \mu h_x^T(x^i)h(x^i) + h_x(x^i)\lambda^i = 0$$

Rewriting this equation in the format of (2) leads to the updating relation:

$$\lambda^{i+1} = \lambda^i + \mu h(x^i)$$

For numerical stability, one of the schemes suggested by [10] is to introduce a small positive scalar ε as follows:

$$\lambda^{i+1} = \lambda^i + \varepsilon\mu h(x^i), \quad 0 < \varepsilon \leq 1 \text{.................................(3)}$$

The complete algorithm of the multiplier method is summarized thus:

(i) Select a sufficiently large scalar μ and an m-dimensional vector λ^0

(ii) Given λ^i, minimizes $\psi^i(x)$:

$$\psi^i(x) = \phi(x) + \frac{1}{2}\mu h^T(x)h(x) + \lambda^{iT} h(x)$$

(iii) Let the optimal solution be x^i; the next estimate of λ is derived from relation (3).

(iv) The minimum x^{i+1} associated with λ^{i+1} will serve as the next approximation to the constrained optimal solution.

2.2 Modified Multiplier Method

While the multiplier method has a clear advantage over the penalty function method due to its requirement for a much smaller penalty coefficient to achieve good results, it also has a significant drawback: many optimization problems must be solved before arriving at the solution for the constrained problem (1). To address this issue, there is a need to develop an optimization procedure that can update the Lagrange multiplier vector as part of the function optimization process:

$$\psi(x) = \phi(x) + \alpha h^T(x)h(x) + \lambda^T h(x)$$

According to [11], the algorithm of the method can be summarized as follows:

(i) Choose the initial values x, λ^0 along with appropriate positive scalars α and $\varepsilon \leq 1$.

(ii) with x^i and λ^i known, the following calculations are performed.

$$g^i = \psi_x(x^i) = \phi_x(x^i) + h_x^T(x^i)\left[h(x^i) + \lambda^i\right]$$

$$p^i = -g^i + \left[\frac{g^{iT} g^i}{g^{i-1T} g^{i-T}}\right] p^{i-1}$$

(iii) The next estimate x^{i+1} is calculated using

$$\psi(x^{i+1}) = \min_{\beta} \psi(x^i + \beta p^i)$$

(iv) Lastly, the Lagrange multipliers are updated using

$$\lambda^{i+1} = \lambda^i + \varepsilon\mu h(x^i)$$

2.3 The Scaled Multiplier Method

Although, the major drawback of the multiplier method is that many optimization problems have to be solved before the solution to the constrained problem was corrected by the method of modified multiplier [11], it was observed that both the multiplier method [10]and [12], and the modified multiplier method using conjugate gradient method as an unconstrained search procedure, repeat a search path for several circles before locating a local optimal points (most especially for optimization problems whose both objective and constrained equation are Nonlinear), which generate much iterations and thereby waste computer time. According to [3] and [13], a good algorithm must not waste computer time.

To create an algorithm that reduces the number of iterations, we applied the conjugate gradient and Newton's methods to nonlinear optimization problems. We utilized the conjugate gradient algorithm for the initial one-dimensional search and then employed the Newton's method for the subsequent iterations. This approach of combining the two unconstrained search methods was successfully tested on several problems.

The complete algorithm of the scaled multiplier method can be summarized as follows:

(i) Select a sufficiently large μ and an m-dimensional vector λ^0.

(ii) Minimize the function

$$\psi^i(x) = \phi(x) + \frac{1}{2}\mu h^T(x)h(x) + \lambda^{iT} h(x)$$

(iii) Compute x^i

(iv) Knowing λ^i,

(v) Find $[H(x^i)]^{-1}$, where H is the inverse Hessian matrix of the function to be optimized.

(vi) Compute the next x^{i+1}

(vii) Let the optimal solution be x^{i+1}, the next estimate of λ is derived from the relation

$$\lambda^{i+1} = \lambda^i + \varepsilon\mu h(x^i).$$

(viii) If the optimal point is not reached, repeat step.

2.4 Analysis of Data

Having summarized the algorithm of the scaled multiplier method (S.M.M), we now present the applications of the method to nonlinear equality constrained problems P_1 and P_2 and the comparison are presented in table P_1 and P_2 respectively. We shall compare our results with the penalty function method (P.F.M) , the classical Lagrangian (C.L), the multiplier method (M.M) and the modified the multiplier method (M.M.M).

Problem P_1

$$\begin{aligned} &Minimize \quad x^2 + 2y^2 + 2xy + 2x + 3y \\ &Sub.to \quad x^2 - y - 1 = 0 \\ &\qquad x_0 = 2, \quad y_0 = 3 \end{aligned}$$

Problem P_2

$$\begin{aligned} &Minimize \quad 2x^2 + 2y^2 + 3xy + 2x + 4y \\ &Sub.to \quad x^2 + y - 2 = 0 \\ &\qquad x_0 = 1, \quad y_0 = 1 \end{aligned}$$

x^*, y^*, λ^* are the values of x, y, λ that gives the optimal value.

ϕ *and* ψ are the optimal values while μ *and* n are the penalty factor and number of iterations respectively.

Table 1: Optimization results table for Problem P_1

Methods	x^*	y^*	λ^*	n	μ	ϕ^*
P.F.M	-0.749858	-0.437482	-	98	130	-1.226236
C.L.M	-0749996	-0.437498	-0.249984	58	-	-1.214637
M.M	-0.749954	-0.437352	-0.249296	12	10	-1.210937

Table 2: Optimization results table for Problem P_2

Methods	x^*	y^*	λ^*	n	μ	ϕ^*
P.F.M	0.26714042	-2.220216	-	115	1000	2.001204
C.L	0.26660841	-2.212161	-5.412884	93	-	1.989714
M.M	0.266094	-2.220160	-2.482795	26	10	1.9079502

3. Conclusion

Table 1 and Table 2 show that the proposed method exhibits high rate of convergence at a lesser number of iterations when compared to the penalty function method (P.F.M) and the Classical Lagrangian (C.L). Although the value of the penalty parameter needed by the method is the same with the multiplier method and the optimal points are almost the same with the multiplier method (M.M) but it converges with lesser number of iterations.

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